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Extending Space-time with Property

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October 2014

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Submitted in fulfillment of the requirements for the Degree of
Doctor of Philosophy

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Paper: Candidate (50%), Author 1 (50%).
Thesis chapter: Candidate (90%), Author 1 (10%).

Paper based on **Chapter 5**: *The General Relativity of Two Properties*

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Abstract

Space-time does an excellent job of describing a "when" and "where" of an event; however it fails to describe "what" was involved. Normally this is achieved by including quantum fields with their own labels and associated properties. Conservation of these properties is then achieved according to what experiment dictates. We propose to add additional coordinates to space-time that describe the "what" which is missing. We choose these coordinates to be complex anti-commuting Lorentz scalars and attach various quantum numbers to them as indicated by experimental observation. With 5 such coordinates we can accommodate all known particles in the Standard Model and can reduce the number of fundamental parameters involved. We also develop formalism to deal with the inclusion of these coordinates into general relativity, and look at the cases of 1 and 2 property coordinates. This results in the Einstein-Maxwell equations for 1 coordinate and the Einstein-Yang Mills equations for 2 coordinates. The Mathematica code used in this thesis, which was essential for the algebraic simplification required, is available from the UTAS digital repository. The documentation for the code is given in Appendix B.

Acknowledgments

Firstly I would like to thank my supervisors Bob and Peter for their continuous guidance over the last few years. Bob's enthusiasm really made this feel like a group project, where every result I got really mattered. The weekly meetings provided motivation but I never felt like I was under high pressure, which made the whole process highly enjoyable and engaging. Next I would like to thank my peers, in particular Courtney, Dave and Vasaant for keeping office life interesting and putting up with my fairly sketchy attendance. Perhaps some days we weren't the most productive, but those days were also the most fun. I would also like to thank the department in general for their support over the years, in particular Simon, Karen and J.D. for always making time to help me when I had administration related issues. Lastly but certainly not least, I would like to thank my family for their unconditional support and care over the years. They were my rock, though their support meant everything always went so smoothly I never needed a rock.

Contents

Chapter 1

Introduction

This thesis outlines an attempt to unify gravity with the other forces as well as explain some aspects of of the Standard Model of particle physics. We do this by introducing complex anti-commuting Lorentz scalar coordinates into space-time. We assign to these coordinates "property" or attribute, for example charge, isospin, colour etc. Particle fields are then obtained by considering products of these coordinates and we can construct superfields by performing series expansions in the property coordinates. Conservation of property in interactions is then achieved by performing Grassmann integration over the property coordinates, rather than by needing to explicitly enforce conservation laws. The introduction of these coordinates results in a space-time-property $\mathbb{Z}_2$ graded manifold. We develop a systematic way of tackling this in terms of general relativity and then apply this to 1 and 2 property coordinates.

The rest of this thesis fills in the details of the above paragraph. Chapter 1 will discuss the Standard Model of particle physics, some other models like Kaluza-Klein theory and supersymmetry and then describe the previous work that has been done on property coordinates. Chapter 2 will more formally introduce the property coordinates and explain how superfield expansions can be done, resulting in mass matrices for particles. Chapter 3 covers the systematic development of general relativity on a $\mathbb{Z}_2$ graded manifold. In Chapters 4 and 5 we look at introducing 1 and 2 property coordinates to space-time and the resulting field equations. The outcome of this is a unification of gravity with electromagnetism and then SU(2) Yang Mills.

1.1 The Standard Model

The Standard Model of particle physics is tried and tested and provides us with the ability to calculate the results of particle physics experiments to high levels of accuracy. There are many textbooks that discuss the Standard Model, for instance Griffiths (2008) provides a solid introduction starting at a basic post-graduate level and Srednicki (2007) is a bit more advanced and covers the basics of quantum field theory. Here we will just discuss some points of the Standard Model that are relevant to the work done in this thesis.

The standard model is based on the gauge group $SU(3) \times SU(2)_L \times U(1)$. Each of these groups in the product have their own associated coupling constant and gauge bosons mediating the corresponding force. The strong nuclear force is mediated by 8 gluons which are the gauge bosons of the $SU(3)$ colour group, the weak nuclear force is mediated by the three $W^{\pm}$ and Z bosons which are gauge bosons of the $SU(2)_L$ group, and finally electromagnetism is mediated by the photon of the $U(1)$ group. There is one other boson under the standard model, the Higgs boson, which will be discussed soon. The fermions are as follows:

6 Leptons:	e	μ	τ	+ antiparticles
6 Neutrinos:	ν_e	ν_μ	ν_τ	+ antiparticles
18 "Up" quarks:	u	c	t	$\times$ 3 colours + antiparticles
18 "Down" quarks:	d	s	b	$\times$ 3 colours + antiparticles

This results in a total of 61 elementary particles. Note the repetition in sets of 3; 3 types of lepton, 3 types of neutrino, 3 types of each of the quarks, this repetition is referred to as generations. Each successive generation is more massive than the previous generation, except possibly for neutrinos where the masses aren't yet known precisely. The latter generations of particles (except neutrinos) decay very quickly down to the lighter first generation, so in everyday life we only come across matter that is composed of electrons, Up quarks and Down quarks.

It should be noted that the L in $SU(2)_L$ refers to the fact that the weak force only acts on left handed particles. This means that the ν_R and $\overline{\nu}_L$ neutrinos, which are both right handed, do not interact at all under the standard model. Originally there was a question as to whether or not they existed, however neutrino oscillations have been observed which firstly indicates neutrinos have mass, and that these sterile non-interacting neutrinos exist as well. However this is under the assumption that neutrinos are Dirac fermions like the other fermions in the standard model. It is also possible that neutrinos are Majorana fermions, where the anti particle is the same as the particle. This removes the need for sterile neutrinos but also removes the distinction between matter and antimatter. The question of whether neutrinos are Dirac or Majorana is still an open question today.

Under the Standard Model, particles get their masses via the Higgs mechanism. The Higgs mechanism makes use of spontaneous symmetry breaking. As a demonstration consider the following Lagrangian for a scalar field ϕ:

$$2\mathcal{L} = (\partial\phi)^2 + \mu^2\phi^2 - \frac{1}{2}\lambda^2\phi^4 \tag{1.1}$$

This Lagrangian has a spontaneously broken $\mathbb{Z}_2$ symmetry, resulting in a non-zero classical expectation value for the scalar field of $\phi = \pm\frac{\mu}{\lambda}$. The Higgs mechanism involves electro-weak spontaneous symmetry breaking by considering a complex scalar Higgs doublet and enforcing a local left handed $SU(2)_L$ gauge symmetry with a Lagrangian similar to the one above, this then results in a massive Higgs field as well as massive gauge vector bosons, the $W^{\pm}$ and Z.

Other fermions can then get their masses via interaction with the Higgs field. This interaction between a Dirac field and a scalar field is called a Yukawa interaction. The Lagrangian for a fermion ψ of mass m with a Yukawa term is:

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi - \alpha\phi\bar{\psi}\psi \tag{1.2}$$

The coupling strength α is different for each type of fermion. If ϕ has a non zero expectation value, for example like the ϕ in Equation 1.1 then it can be rewritten in terms of shifted field $\eta = \phi + \frac{\mu}{\lambda}$. The Yukawa term in Equation 1.2 then becomes a mass term plus an interaction term, resulting in the following Lagrangian for the fermionic field:

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - \alpha\frac{\mu}{\lambda}\bar{\psi}\psi - \alpha\eta\bar{\psi}\psi \tag{1.3}$$

The mass term from Equation 1.2 has been dropped, as it is now replaced by the term resulting from the expectation value of the scalar field ϕ. Note however that while this gives us a mechanism for giving fermions mass, since each fermion has its own coupling α to the scalar Higgs field it doesn't allow us to calculate the masses of fermions or reduce the number of free parameters. When considering the full Standard Model Lagrangian we have 6 quarks, 3 leptons, 3 neutrinos and the Higgs Boson each of which has its own mass; there is also the CKM quark mixing matrix and the MNS neutrino mixing matrix each of which is described by 4 parameters and finally 3 coupling constants for electromagnetism and the strong/weak nuclear forces. This results in at least 24 parameters that have to be determined experimentally.

The Standard Model is incapable of explaining why we observe 3 generations of particles, nor does it give any hints as to whether there are more. There are also no candidates for Dark matter in the Standard Model, which is the current leading explanation for galactic rotation curves. The large number of parameters along with the open questions regarding the Standard Model leads many to think there must be some underlying theory that produces the Standard Model. Gravity is not included in the Standard Model either. Though this poses its own set of challenges, there is no solid theory of quantum gravity yet. Unifying gravity with the other forces has been attempted in the past and one of the most notable attempts is that of Kaluza-Klein theory.

1.2 Kaluza-Klein theory

Kaluza-Klein theory involves attaching an extra spatial dimension to space-time in an attempt to unify gravity and electromagnetism. The original papers by Kaluza and Klein in 1921 and 1926 are in German, however Overduin and Wesson (1997) provides an introduction and review of Kaluza-Klein theories. The additional spatial dimension is typically compactified into a small circle to explain why we observe 4 space-time dimensions, and also why the 5th dimension does not directly appear in the laws of physics. The starting point for Kaluza-Klein is to choose a form for the metric in 5 dimensions as follows:

$$G_{MN} = \begin{pmatrix} g_{mn} + k^2\phi^2 A_m A_n & k\phi^2 A_m \\ k\phi^2 A_n & \phi^2 \end{pmatrix} \tag{1.4}$$

We have used some slightly non-standard notation here, to be consistent with the notation used later in this thesis. The 5 dimensional metric is written as G_{MN}, where capital letters M, N, etc run over 1 to 5. The 4 dimensional metric is written as g_{mn}, where lower case letters m, n, etc run over the standard space-time dimensions 1 to 4. ϕ is a scalar field and A_m is the gauge field for electromagnetism with scaling factor k. We take this metric and then consider the gravitational part of the Einstein-Hilbert action:

$$S = \frac{1}{2\kappa} \int \sqrt{-G}\, R\, d^5x \tag{1.5}$$

where R is the 5 dimensional Ricci "superscalar" and $\sqrt{-G}$ is the volume element on the 5 dimensional manifold. If we consider the case where $\phi =$ constant, which is essentially what will be done later in this thesis, variation of this action produces the following field equations:

$$R_{mn} - \frac{1}{2} G_{mn} R = \frac{1}{2} k^2 \phi^2 T_{mn} \tag{1.6}$$

$$F_{mn}{}^{;n} = 0 \tag{1.7}$$

where $T_{mn} = F_m{}^s F_{sn} - \frac{1}{4} g_{mn} F_{kl} F^{kl}$ is the electromagnetic stress-energy tensor. The first of these equations is the Einstein field equations for electromagnetism. Without explicitly including gauge fields in the action we now have the gauge fields appearing correctly as a part of the geometry. The second of these equations is the Maxwell equations in curved space. The result of all this is that in the case of pure electromagnetism we now have the equations that govern both gravity and electromagnetism produced by considering a single action, in effect unifying those forces. Work has also been done on higher dimensional Kaluza-Klein theories in an attempt to unify the other forces with gravity as well. One of the issues with Kaluza-Klein theories is the infinite number of Fourier modes in the 5th coordinate, potentially resulting in an infinite number of particles with increasing mass. Kaluza got around this by only considering the ground modes, but this causes problems for higher dimensional Kaluza-Klein theories that make use of compactification.

1.3 Georgi-Glashow SU(5) model

The Standard Model is a gauge theory based on the group $SU(3) \times SU(2)_L \times U(1)$, with 3 separate coupling constants for the $SU(3)$ colour force, $SU(2)_L$ weak force, and $U(1)$ electromagnetism. Georgi and Glashow (1974) proposed a model to unify these forces with 1 coupling constant under an $SU(5)$ gauge group which contains the $SU(3) \times SU(2)_L \times U(1)$

group of the Standard Model. The Standard Model we observe is due to a spontaneous symmetry breaking of the overall $SU(5)$ group, resulting in the Standard Model for low energies. Particles are fitted into various representations of $SU(5)$, though this model also predicts some other particles like super-heavy coloured vector bosons. The presence of these bosons allows the proton to decay, which is disallowed under the Standard Model. The fact that they are super-heavy means the lifetime of the proton is quite long; however experiments have been performed to measure the decay rate of the proton and no decays have been observed. Clark (2007) for instance looked for proton decays in 4 different channels and found the lifetime of the proton has to be at least order 10^{33} years, which essentially rules out the original Georgi-Glashow model. The model is still worth mentioning however as it is used as the basis for several modern theories which have greatly increased proton half-lives. For a review of grand unified theories, including $SU(5), SO(10)$ and $SU(2) \times SU(2) \times SU(4)$ see Baez and Huerta (2010).

1.4 Extended General Relativity

Attempts to unify gravity with the other forces in a manner similar to Kaluza-Klein by extending general relativity on a higher dimensional supermanifold have been studied quite extensively. This work makes extensive use of concepts from topology, like fibre bundles on differential manifolds. Trautman (1970) provides an introduction to these ideas aimed at physicists, and then suggests how they can be used to extend Kaluza-Klein theory to non abelian groups. These ideas are used by several authors to attempt to unify the non abelian gauge forces with gravity. For instance Cho (1975) extends Kaluza-Klein type unifications by using non abelian gauge fields, and produces an Einstein-Hilbert action which results in the unification of gravity with a non abelian gauge field plus a cosmological constant. Tabensky (1976) takes a slightly different approach, but produces a similar result. Tabensky (1976) also notes that they have spin-1 bosons of the Yang-Mills type and spin-0 bosons, which are coupled to spin-2 gravitons, but no fermions. One way to introduce fermions in a theory like this is by including transformations that mix fermions and bosons, this is what leads to supersymmetry.

1.5 Supersymmetry

Supersymmetry is currently one of the leading areas of research into physics beyond the Standard Model. Zee (2010) provides an introductory chapter on supersymmetry; some of the basic concepts will be outlined here. The underlying idea behind supersymmetry is to introduce a symmetry between bosons and fermions. Currently we do not observe any such symmetries in nature, but this doesn't necessarily mean they don't exist at some higher energy our accelerators have not managed to reach yet. The first step is to define some supersymmetry generators $Q_{\mathcal{N}\alpha}$, which take us from bosonic fields ϕ to fermionic fields ψ_α.

In extended supersymmetry there are $\mathcal{N}$ of these generators. Increasing $\mathcal{N}$ results in more supersymmetry, which constrains the field content. For $\mathcal{N} > 8$ there are massless fields of spin > 2 that are produced, which cause issues for the theory. As a result of this $\mathcal{N} = 8$ is considered to be maximally supersymmetric. We will now look at $\mathcal{N} = 1$ supersymmetry, for which we have the following commutation relations:

$$[P^\mu, Q_\alpha] = 0 \tag{1.8}$$
$$[J^{\mu\nu}, Q_\alpha] = -i(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta \tag{1.9}$$
$$[J^{\mu\nu}, \bar{Q}^{\dot{\alpha}}] = -i(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{Q}^{\dot{\beta}} \tag{1.10}$$
$$\left\{Q_\alpha, \bar{Q}_{\dot{\beta}}\right\} = 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu \tag{1.11}$$
$$\{Q_\alpha, Q_\beta\} = \left\{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\right\} = 0 \tag{1.12}$$

These operators are independent of space-time coordinates, and so commute with the momentum operator P^μ, giving the first commutation relation. They transform as Weyl spinors, which gives the second and third commutation relations with the Lorentz generators $J^{\mu\nu}$. The last two lines give the supersymmetry algebra, which essentially comes out as the only possibility for the Grassmann Q_α.

The momentum operator can be thought of as generating translation in space-time. The corresponding idea for supersymmetry generators would be to generate translation in some Grassmann coordinates θ^α and $\bar{\theta}^{\dot{\beta}}$. This leads to the concept of a superspace, with coordinates $(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\beta}})$. The x^μ are the standard space-time coordinates and are bosonic, the θ^α and $\bar{\theta}^{\dot{\beta}}$ are fermionic and transform as 2 component Weyl spinors. Superfields can then be constructed on this superspace, $\Phi(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\beta}})$.

One can also define operators D_α and $\bar{D}_{\dot{\beta}}$ which anticommute with Q_α and $\bar{Q}_{\dot{\beta}}$. A superfield that satisfies $\bar{D}_{\dot{\beta}}\Phi = 0$ is called a chiral superfield. We can then define $y^\mu = (x^\mu + i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}})$, which satisfies $\bar{D}_{\dot{\beta}}y^\mu = 0$. Thus if we have a superfield $\Phi(y^\mu, \theta^\alpha)$, that depends only on y and θ it will automatically satisfy $\bar{D}_{\dot{\beta}}\Phi = 0$ and hence be a chiral superfield. When constructing $\Phi(y, \theta)$ we perform a power series expansion in θ. As these are two component Grassmann objects the highest power possible is $\theta\theta$, the power series terminates here. Thus $\Phi(y, \theta)$ takes the form:

$$\Phi(y, \theta) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \tag{1.13}$$

This superfield contains a Weyl fermion field ψ as well as two complex scalar fields, ϕ and F. This superfield can be used to construct an action invariant under supersymmetry. One example action is as follows:

$$S = \int d^4x \left(\left[\Phi^\dagger\Phi\right]_D - [\frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3]_F + \left([\frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3]_F\right)^\dagger \right) \tag{1.14}$$

where $[X]_D$ refers to only using the $\theta\theta\bar{\theta}\bar{\theta}$ coefficient, $[X]_F$ refers to only using the $\theta\theta$ coefficient. This action can then be used to find the equations of motion for the fields and their interactions.

While supersymmetry is one of the leading areas of research into physics beyond the Standard Model, we have not yet discovered any experimental evidence for it. The data from the LHC has been used to search for supersymmetric partners to known particles but so far nothing has come up, forcing theorists to keep raising the mass of the lightest superpartner in their theories. Even those involved in the research of supersymmetry are starting to ask at what point should we give up on it. The fact that supersymmetry hasn't worked out so far means searching for alternative theories warrants serious consideration. This thesis is an attempt to develop one of these theories, which borrows some ideas from the other theories discussed here but is different to anything done before.

1.6 Property coordinates

We now reach the model considered in this thesis. The following section will consist of two aspects, first describing the historical development of the theory and then secondly describing its state before the commencement of this thesis. This will allow us to then demonstrate the original work carried out in this thesis extending work on the model. The specific details of the theory will be described in later chapters, so as to allow a consistent build up of the concepts involved.

One of the first papers considering the addition of Grassman coordinates to spacetime is Arnowitt and Nath (1976). Using some of the ideas of supersymmetry Grassman coordinates are included in spacetime and local gauge symmetry is applied. By considering the Einstein-Hilbert Lagrangian, this results in unification of gravity with electromagnetism. The work by Arnowitt and Nath (1976) is later expanded upon by Delbourgo and Zhang (1988a), Delbourgo et al. (1988) and Delbourgo and Zhang (1988b). These papers consider including 5 complex Grassmann variables ζ to the standard 4 of space-time to produce the Standard Model and also the possibility of unifying gravity and the other forces. Particles are represented by monomials in these coordinates, with the observed particles of the Standard Model accommodated easily. There were however a reasonably large excess of particle states, some of which produced anomalies. Delbourgo and White (1990) suggested a method of cutting down on these states via self-duality. This was implemented by Delbourgo et al. (1991) to reduce the number of particle states for the 5 coordinate model; the anomalies could be dealt with while still having enough room for the Standard Model. Delbourgo et al. (1993) considered in detail the various Lie algebras and representations that could be constructed from N Lorentz scalar Grassmann variables and their derivatives. Two models with N = 5 were considered and by applying duality restrictions both were made anomaly free. The first took odd monomials (products of an odd number of Grassmann coordinates) to be right-handed chiral spinors, while even monomials were left-handed spinors.

Delbourgo (2006a) associated the 5 Grassmann coordinates with specific quantum numbers, like charge and fermion number, making them carriers of the property of particles. The assignments are as follows:

Property coordinate:	ζ^0	ζ^1	ζ^2	ζ^3	ζ^4
Charge (units of e):	0	1/3	1/3	1/3	-1
Fermion number:	1	-1/3	-1/3	-1/3	1
Colour:	None	Red	Green	Blue	None

Fermions are made up from odd monomials in the property coordinates and bosons are made up from even monomials. This arrangement preserves spin-statistics. A self-duality restriction is then applied to reduce the number of particle states and to eliminate anomalies. The resulting fermions include 3 up-type quarks, 8 down-type quarks, 6 charged leptons, 4 neutrinos and a corresponding set of antiparticles. The fermions of the Standard Model are accommodated, along with some extra particles that have not been observed yet. There are also 9 Higgs-like neutral scalars. These could in theory impart masses to the fermions via their 9 expectation values, resulting in less parameters than the Standard Model. Superfields can also be constructed: $\Phi(x, \zeta, \bar{\zeta})$ is a superfield describing bosons, which contains the even monomials in the property coordinates that survive the duality restrictions; similarly $\Psi(x, \zeta, \bar{\zeta})$ is a superfield describing fermions and contains the odd monomials of the property coordinates.

Unification of gravity with electromagnetism was also attempted in this paper, by considering general relativity with 1 property coordinate. However rather than appending one complex anti-commuting Lorentz scalar ζ and its conjugate $\bar{\zeta}$ to space-time, two real coordinates (ξ, η) were considered instead. Two supermetrics G_{MN} were considered, the first includes curvature in the property sector but no gauge fields:

$$\begin{pmatrix} G_{mn} & G_{m\xi} & G_{m\eta} \\ G_{\xi n} & G_{\xi\xi} & G_{\xi\eta} \\ G_{\eta n} & G_{\eta\xi} & G_{\eta\eta} \end{pmatrix} = \begin{pmatrix} g_{mn}(1 + if\xi\eta) & 0 & 0 \\ 0 & 0 & -i\Lambda^2(1 + ig\xi\eta) \\ 0 & i\Lambda^2(1 + ig\xi\eta) & 0 \end{pmatrix} \tag{1.15}$$

The Einstein-Hilbert action produced from this metric included the standard gravitational action plus a cosmological constant. The other supermetric considered involved gauge fields but no property sector curvature:

$$\begin{pmatrix} G_{mn} & G_{m\xi} & G_{m\eta} \\ G_{\xi n} & G_{\xi\xi} & G_{\xi\eta} \\ G_{\eta n} & G_{\eta\xi} & G_{\eta\eta} \end{pmatrix} = \begin{pmatrix} g_{mn}(1 + if\xi\eta) + 2i\Lambda^2\xi A_m A_n\eta & i\Lambda^2 A_m\xi & i\Lambda^2 A_m\eta \\ i\Lambda^2 A_n\xi & 0 & -i\Lambda^2 \\ i\Lambda^2 A_n\eta & i\Lambda^2 & 0 \end{pmatrix} \tag{1.16}$$

This supermetric produces an Einstein-Hilbert action containing the gravitational curvature plus the electromagnetic Lagrangian. The inclusion of the gauge fields in the space-property section of the supermetric is similar to the Kaluza-Klein model, with a similar result as well, unifying the gravitational and electromagnetic forces.

Some formalism for considering general relativity on a graded manifold was also developed in this paper. However there were some small self-consistency issues that will be elaborated on in Chapter 3 of this thesis.

Delbourgo (2006b) considered the Yukawa interactions present in model from Delbourgo (2006a). The 9 neutral Higgs fields are assumed to have independent expectation values, resulting in an expectation value for the Higgs superfield $\langle\Phi\rangle$. Flavor mixing and mass matrices are produced by considering a Yukawa term of the form:

$$\int d^5\zeta d^5\bar{\zeta}\ \bar{\Psi}\langle\Phi\rangle\Psi \tag{1.17}$$

Terms from this expansion can then be selected to give flavor mixing matrices, 8×8 for down quarks, 6×6 for leptons etc. An attempt was made to give numerical values for the 9 Higgs expectation values to see if a sensible mass spectrum could be obtained. While the quark masses were satisfactory, the mixing matrices didn't work out so well, nor the masses of the light leptons. The possible space of values was not searched completely, so there is still room for the model to work out. If the duality restriction were dropped from the Higgs superfield Φ then there would be 18 possible Higgs expectation values to play with, however this is nearly as many parameters that the Standard Model has in the first place!

1.7 Original Thesis work

This thesis updates, verifies and builds on the previous work regarding property coordinates. Here we will outline the original research conducted as a part of this thesis.

The most significant contribution is the development of Mathematica code designed to deal with the large amount of algebra required. Previous work on property coordinates had been done by hand, which limited what could be achieved. Many of the results found in this thesis would not have been possible without the significant amount of time invested into developing Mathematica code. The code itself is available from the UTAS library digital repository and is discussed in Appendix B.

The 9 Higgs fields mentioned in Delbourgo (2006b) have their expectation values treated as independent variables, when in theory they should be produced by spontaneous symmetry breaking. In this Chapter 2 we perform the algebra to determine the spontaneous symmetry breaking for the Higgs fields. For a renormalisable theory this requires evaluation of Φ^2, Φ^3 and Φ^4, where Φ is the superfield expansion containing the 9 Higgs fields. These are used to produce the Higgs potential which is then minimised to get conditions on the expectation values of the Higgs fields. The number of parameters in the model is greatly reduced by this process, as all fermion masses are then dependent on the 3 parameters in the Higgs Lagrangian instead of the 9 Higgs expectation values.

Delbourgo (2006a) included a section outlining how to deal with general relativity on a graded manifold. This was used as the starting point for the work done in Chapter 3 of this thesis, however there are some issues with the formalism developed in that paper. The formalism is re-derived in Chapter 3, with several checks performed for self-consistency.

That paper also looked at General Relativity with 1 property coordinate. However instead of using a complex Grassmann coordinate and its conjugate, 2 real coordinates were used. Since the algebra involved was so difficult to perform by hand two separate cases were considered, a supermetric with curvature in property only and another with gauge fields only. These were also only determined in the case of Minkowski space-time, the gravitational factors were then included by assuming the tensors had to be generally covariant. Chapter 4 considers the case of 1 property coordinate included in space-time, but does so with complex Grassmann coordinates, developing the formalism required. A general supermetric is used that includes both curvature in the property sector and gauge fields at the same time. Using the Mathematica code developed, the Einstein-Hilbert action is then determined in the general case of curved space-time. The field equations are also considered for both space-time and the gauge field. This is all repeated in Chapter 5, except with 2 property coordinates. The additional formalism required to deal with non-abelian gauge fields as well as the supermetric, resulting Einstein-Hilbert action and field equations are all included.

The vast majority of the remaining sections of this thesis represent original research. The parts that are repeated from Delbourgo (2006a) and Delbourgo (2006b) are re-derived to check the results using Mathematica.

Chapter 2

Property coordinates, Superfields

Space-time forms a solid framework in which we can describe the position and time of events and interactions. We can also describe the momentum and energy transfers that occur in these events, but when it comes to describing the properties of the particles and fields involved space-time falls short. Usually we just assign some external labels to specify what particles are involved in any given interaction, but what if we didn't have to do this? The focus of this thesis is to see what happens when extra coordinates are appended to space-time in an attempt to describe particle properties in one coherent framework. Note that it is assumed the reader has at least a passing familiarity with the concepts of superspaces and superfield expansions. For background reading any good supersymmetry textbook should suffice, for example Wess and Bagger (1992) provides an introduction to these ideas and Weinberg (2000) provides a more involved explanation.

2.1 Property model

The model presented in this thesis is constructed as follows:

1. We attach to space-time N complex property coordinates $\zeta = (\zeta^1, \zeta^2, ..., \zeta^N)$ and their complex conjugates $\bar{\zeta} = (\zeta^{\bar{1}}, \zeta^{\bar{2}}, ..., \zeta^{\bar{N}})$ to form a $4 + 2N$ dimensional superspace $X = (x, \zeta, \bar{\zeta})$.

2. We choose these coordinates to be anti-commuting, $\zeta^\mu \zeta^\nu = -\zeta^\nu \zeta^\mu$. This results in a graded superspace; the 4 dimensional space-time part is even graded and the $2N$ dimensional property sector is odd graded. The property sector is an $\mathrm{Sp}(2N)$ group, for a detailed discussion of the group structure see Delbourgo et al. (1993).

3. Anti-commutativity also this ensures that superfield expansions will terminate with a finite number of terms since $\zeta^\mu \zeta^\mu = -\zeta^\mu \zeta^\mu$, which is true if and only if $\zeta^\mu \zeta^\mu = 0$. So for example a superfield expansion with one property coordinate would take the form: $S(x) = a(x) + b(x)\zeta + c(x)\bar{\zeta} + d(x)\bar{\zeta}\zeta$, with a, b, c and d being space-time dependent particle fields.

4. Any term in a superfield expansion can only contain one of each ζ or $\bar{\zeta}$, this is also where the idea of a property comes in, a particle can only either have or not have a particular property associated with a given ζ or $\bar{\zeta}$. Add the quantum numbers of individual ζ and $\bar{\zeta}$ to get the overall quantum numbers.

5. We choose to have the conjugates providing the opposite property of their type. This leads naturally to anti-particles with complex conjugation being associated with charge conjugation, swapping between a particle and its anti-particle. Since a property and its complex conjugate cancel each other out this also produces generations of particles, where the same quantum numbers are present but extra property/conjugate pairs change the coupling to the Higgs field and hence the masses.

6. An odd number of property coordinates is overall anti-commuting, while an even number of property coordinates commutes. To satisfy spin-statistics we associate odd numbers of property coordinates with fermionic particle fields, and even ones with bosonic fields.

7. We choose these coordinates to be Lorentz scalar, this separates our scheme from supersymmetry and greatly simplifies the algebra when we begin to consider general relativity on the superspace in later chapters.

We now want to know how many coordinates are required to include all known particles from the standard model. This is done by considering superfield expansions; which are power series expansions in the property coordinates, where each term represents a particle field. It is clear there must be at least 3 such coordinates, one for each colour to model quarks. We can assign colour to 3 property coordinates as follows:

Coordinate	ζ^1	ζ^2	ζ^3
Colour	Red	Green	Blue

(2.1)

This is sufficient to model the strong force. However to produce a colourless fermionic particle, like an electron or neutrino with an odd number of property coordinates, the only possible combination is to include all three of them, $\zeta^1\zeta^2\zeta^3$. This means we don't have the ability to produce both electrons and neutrinos, so we need at least 4 coordinates. The questions of whether 4 is enough is a bit more subtle, but if you assign charge and colour to 4 coordinates it isn't possible to produce three generations of electron, neutrino, down quark and up quark made up of *odd* numbers of property coordinates. This means we are forced to use 5 property coordinates ($\zeta^0, \zeta^1, \zeta^2, \zeta^3, \zeta^4$ and conjugates $\zeta^{\bar{0}}, \zeta^{\bar{1}}, \zeta^{\bar{2}}, \zeta^{\bar{3}}, \zeta^{\bar{4}}$), which easily includes all the particles of the standard model but also many more. Let us count the number of property coordinates ζ that make up a term in the superfield expansion and call that p, then count the number of conjugate coordinates $\bar{\zeta}$ and call that q. We can then group terms in the superfield expansion by their corresponding (p, q) pair. The number of possible particle states for a given p and q is as follows:

Number of terms in superfield expansion of type (p,q)

$$\begin{array}{cc|cccccc} & & & & & p= & & \\ & & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline & 0 & 1 & 5 & 10 & 10 & 5 & 1 \\ & 1 & 5 & 25 & 50 & 50 & 25 & 5 \\ & 2 & 10 & 50 & 100 & 100 & 50 & 10 \\ q= & 3 & 10 & 50 & 100 & 100 & 50 & 10 \\ & 4 & 5 & 25 & 50 & 50 & 25 & 5 \\ & 5 & 1 & 5 & 10 & 10 & 5 & 1 \end{array} \tag{2.2}$$

For example this means there are 50 possible superfield terms and hence particles of type (2,1), that is with 2 ζ and 1 $\bar{\zeta}$ present.

2.2 Notation

Before considering our model further we first need to establish some notational convention. Products of ζ and their conjugates $\bar{\zeta}$ will appear frequently in this chapter. To reduce the amount of space taken up, and to allow a clearer picture of what is going on, a shorthand notation will be used. First the products are arranged into a canonical order; with ζ^{μ}'s arranged into increasing order of μ followed by the conjugate coordinates $\zeta^{\bar{\mu}}$ arranged in increasing order of $\bar{\mu}$. This arrangement can always be done as the coordinates anti-commute with each other, with a sign produced depending on the sign of the permutation required to form the canonical order. For example:

$$\begin{aligned} \text{Unordered Product} &\quad \text{Ordered Product} \\ \zeta^2\zeta^3\zeta^1 &= \zeta^1\zeta^2\zeta^3 \\ \zeta^{\bar{1}}\zeta^2\zeta^3\zeta^4 &= -\zeta^2\zeta^3\zeta^4\zeta^{\bar{1}} \\ \zeta^{\bar{4}}\zeta^1\zeta^{\bar{3}}\zeta^{\bar{2}}\zeta^{\bar{1}} &= -\zeta^1\zeta^{\bar{1}}\zeta^{\bar{2}}\zeta^{\bar{3}}\zeta^{\bar{4}} \end{aligned} \tag{2.3}$$

The repeated ζ and $\bar{\zeta}$ do not provide any useful extra information, so the product can be written in shorthand removing these repeated symbols. For example:

$$\text{Full product} \quad \text{Shorthand} \tag{2.4}$$

$$\zeta^1\zeta^2\zeta^3 = \zeta^{123} \tag{2.5}$$

$$\zeta^2\zeta^3\zeta^4\zeta^{\bar{1}} = \zeta^{234\bar{1}} \tag{2.6}$$

$$\zeta^1\zeta^{\bar{1}}\zeta^{\bar{2}}\zeta^{\bar{3}}\zeta^{\bar{4}} = \zeta^{1\bar{1}\bar{2}\bar{3}\bar{4}} \tag{2.7}$$

This process can always be reverted by simply filling in the missing ζ symbols. Finally integration over the property coordinates is done like any Grassmann coordinates, $\int \zeta d\zeta = 1$,

$\int d\zeta = 0, \int \bar{\zeta} d\zeta = 0, \int \bar{\zeta}\zeta d\zeta d\bar{\zeta} = 1$ etc. See Berezin (1987); DeWitt (1984) for more background on Grassmann coordinates.

2.3 Charge conjugation

We take the charge operator ^{c} to act like a Hermitian conjugate on the property coordinates. The order of the property coordinates is reversed and then they are swapped between $\zeta \leftrightarrow \bar{\zeta}$. This changes between particles and anti-particles as the conjugate property coordinates have the opposite quantum numbers. For example $(\zeta^{123})^c = (\zeta^3)^c(\zeta^2)^c(\zeta^1)^c = \zeta^{\bar{3}}\zeta^{\bar{2}}\zeta^{\bar{1}} = -\zeta^{\bar{1}\bar{2}\bar{3}}$. In terms of Table 2.2 charge conjugation is represented by a swapping of p and q, so for a particle state belonging to the group (2,1) its anti-particle will belong to (1,2). This produces a symmetry about the $p = q$ line on the table, halving the number of fermions we have to consider. To further reduce the number of particle states we will apply another symmetry condition we call anti selfduality.

2.4 Self duality

Following the work done by Delbourgo et al. (1991) and Delbourgo et al. (1993), we can impose a symmetry restraint on the possible particle states using the dual operator. The dual operator is effectively a reflection in the cross diagonal of Table 2.2, taking (p, q) to $(5 - q, 5 - p)$. It can be constructed from the 5-dimensional Levi-Civita as follows:

$$(\zeta^{a_1 \ldots a_p \bar{b}_1 \ldots \bar{b}_q})^\times = \varepsilon^{a_1 \ldots a_p a_{p+1} .. a_5} \varepsilon^{\bar{b}_1 \ldots \bar{b}_q \bar{b}_{q+1} \ldots \bar{b}_5} \zeta^{b_{q+1} \ldots b_5 \bar{a}_{p+1} \ldots \bar{a}_5} / ((5-p)!(5-q)!). \tag{2.8}$$

Here rather than the standard Einstein summation over repeated "up" and "down" indices we have summation over repeated indices and their conjugates (sum over a_i and $\bar{a}_i$, b_j and $\bar{b}_j$). The factor of $1/(5-p)!(5-q)!$ is for counting, as there are $p!$ ways of permuting p indices. The following table lists general forms of duals for a few example (p, q) types as well as some specific examples:

Product	Type	Dual	Dual type
1	(0,0)	$\varepsilon^{ijklm}\varepsilon^{\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}}\zeta^{abcde\bar{i}\bar{j}\bar{k}\bar{l}\bar{m}}/(5!5!) = \zeta^{01234\bar{0}\bar{1}\bar{2}\bar{3}\bar{4}}$	(5,5)
ζ^i	(1,0)	$\varepsilon^{ijklm}\varepsilon^{\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}}\zeta^{abcde\bar{j}\bar{k}\bar{l}\bar{m}}/(4!5!)$	(5,4)
$\zeta^{ij\bar{a}}$	(2,1)	$\varepsilon^{ijklm}\varepsilon^{\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}}\zeta^{bcde\bar{k}\bar{l}\bar{m}}/(3!4!)$	(4,3)
$\zeta^{ijk\bar{a}\bar{b}}$	(3,2)	$\varepsilon^{ijklm}\varepsilon^{\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}}\zeta^{cde\bar{l}\bar{m}}/(2!3!)$	(3,2)
ζ^{123}	(3,0)	$\varepsilon^{12304}\varepsilon^{\bar{0}\bar{1}\bar{2}\bar{3}\bar{4}}\zeta^{01234\bar{0}\bar{4}} =$ -$\zeta^{01234\bar{0}\bar{4}}$	(5,2)
$\zeta^{234\bar{1}}$	(3,1)	$\varepsilon^{23401}\varepsilon^{\bar{1}\bar{0}\bar{2}\bar{3}\bar{4}}\zeta^{02340\bar{1}} =$ -$\zeta^{023 4\bar{0}\bar{1}}$	(4,2)
$\zeta^{1\bar{1}\bar{2}\bar{3}\bar{4}}$	(1,4)	$\varepsilon^{10234}\varepsilon^{\bar{1}\bar{2}\bar{3}\bar{4}\bar{0}}\zeta^{0\bar{0}\bar{2}\bar{3}\bar{4}} =$ -$\zeta^{0\bar{0}\bar{2}\bar{3}\bar{4}}$	(1,4)
$\zeta^{12\bar{3}\bar{4}}$	(2,2)	$\varepsilon^{12034}\varepsilon^{\bar{3}\bar{4}\bar{0}\bar{1}\bar{2}}\zeta^{012\bar{0}\bar{3}\bar{4}} = \zeta^{012\bar{0}\bar{3}\bar{4}}$	(3,3)

Note that we use the convention that $\varepsilon^{01234} = \varepsilon^{\bar{0}\bar{1}\bar{2}\bar{3}\bar{4}} = 1$. It is important that the dual operator preserves property, for example the product $\zeta^{12\bar{3}\bar{4}}$ is taken to the product $\zeta^{012\bar{0}\bar{3}\bar{4}}$, which can be written as $\zeta^{12\bar{3}\bar{4}}\zeta^{0\bar{0}}$, the $\zeta^{0\bar{0}}$ part does not change the property as it is a conjugate pair. Since the dual operator preserves property it can be used to place a symmetry restriction on the superfield expansions. We now require that the superfield expansions are anti selfdual. So for instance $\zeta^{12\bar{3}\bar{4}}$ would be grouped with its dual to form a single term: $(\zeta^{12\bar{3}\bar{4}} - \zeta^{012\bar{0}\bar{3}\bar{4}})$ which under the dual operator changes sign. This effectively cuts the number of possible particle states from Table 2.2 in half. Terms that are selfdual are eliminated by this process, for instance the product ζ^{01234} goes to itself under the dual operation, so to maintain anti selfduality this term becomes zero. Similarly terms of the form $\zeta^{0123\bar{4}}$ or $\zeta^{012\bar{3}\bar{4}}$ where there are no conjugate pairs are also eliminated.

Number of terms in superfield expansion of type (p, q) after symmetry reduction

		$p=$ 0	1	2	3	4	5
	0	1	5	10	10	5	0
	1	$*$	25	50	50	10	$\times$
	2	$*$	$*$	100	45	$\times$	$\times$
$q=$	3	$*$	$*$	$*$	$\times$	$\times$	$\times$
	4	$*$	$*$	$*$	$*$	$\times$	$\times$
	5	$*$	$*$	$*$	$*$	$*$	$\times$

(2.9)

The * are the anti-particles and the $\times$ are equivalent under the anti selfduality condition. Delbourgo et al. (1993) go into more detail regarding how the duality and charge operators arise from the algebra of Lie group automorphisms, though the actual model we use is similar to Delbourgo (2006b).

2.5 Five coordinate model

Quantum numbers We can now assign quantum numbers to our property coordinates ζ and $\bar{\zeta}$

Property	ζ^0	ζ^1	ζ^2	ζ^3	ζ^4
Charge	0	$-1/3$	$-1/3$	$-1/3$	1
Colour	None	Red	Green	Blue	None
Lepton Number	1	0	0	0	-1
Fermion Number	1	1/3	1/3	1/3	-1
Similar to	Neutrino	Down Quark	Down Quark	Down Quark	Anti Electron

Property	$\zeta^{\bar{0}}$	$\zeta^{\bar{1}}$	$\zeta^{\bar{2}}$	$\zeta^{\bar{3}}$	$\zeta^{\bar{4}}$
Charge	0	+1/3	+1/3	+1/3	−1
Colour	None	AntiRed	AntiGreen	AntiBlue	None
Lepton Number	−1	0	0	0	1
Fermion Number	−1	−1/3	−1/3	−1/3	1
Similar to	Anti Neutrino	Anti Down	Anti Down	Anti Down	Electron

(2.10)

The charge listed is in units of the charge of a proton. The above assignment of quantum numbers is based on SU(5) and SO(10) grand unified theories, see Georgi and Glashow (1974) and Georgi (1975) for the original papers, and neatly produces all the observed particles in the standard model. By requiring that the superfield expansions are anti selfdual some "bad" quantum states are eliminated, for instance $\zeta^{0123\bar{4}}$ which would have been a particle with fermion number 3 and charge −2. We can list where the standard model fermions appear in Table 2.9, writing: L for charged leptons, N for neutrons, D for down quarks, U for up quarks and including ^{c} for charge conjugates.

Standard model fermions by (p, q) pair

		$p=$ 0	1	2	3	4	5
	0		N_1, L_1^c, D_1		L_5, D_7^c, U_3		
	1	$*$		$N_{2,3}, L_{2,3}^c, D_{2,3,4}, U_1^c$		L_6, D_8^c, U_4	
	2		$*$		$L_4^c, N_4, D_{5,6}, U_2^c$		$\times$
$q=$	3	$*$		$*$		$\times$	
	4		$*$		$*$		$\times$
	5	$*$		$*$		$*$	

(2.11)

The above table lists the particle states similar to those of standard model particles with the correct colour, charge and fermion number. We have 4 generations of Neutrino, 6 generations of charged Leptons, 8 generations of Down quarks and 4 generations of Up quarks. It appears this breaks the symmetry between the Leptons, as we no longer have a neutrino paired up with every charged lepton, but 2 of the charged "leptons" are produced by the combination ζ^{123} which makes them have a lepton number of zero. We also have 8 generations of Down quarks which is excessive, but required to allow for enough generations of Up quark. There are many other types of exotic particles present. Overall including both standard model and exotic particles, but not anti-particles, we have the following fermions:

Charge	−4/3	−1	−2/3	−1/3	0	1/3	2/3	1	4/3	5/3
Count	6	20	12	6	20	30	12	2	6	6

(2.12)

This extra abundance of charged particles does provide a test for our model, by considering the decay of the Higgs boson into two photons via a quark loop (Figure 2.1). The rate of

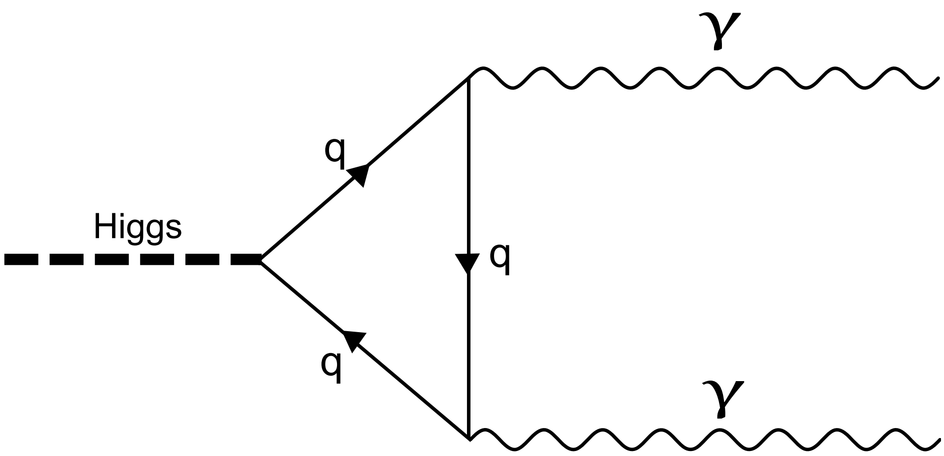


Figure 2.1: Higgs decay into two photons via quark loop

this decay is dependent on the charge squared of the quark in the loop and the mass of the quark involved (Egede 1998). Only the top quark is heavy enough to provide any significant contribution under the standard model, our model however admits new particles that we haven't observed yet, so they must be heavier than the top. If we assume these new particles are very heavy, the mass dependence becomes weak and we can simply sum the squares of the charges to get an estimate of the expected decay rate. Performing this for our model gets a back of the envelope estimate of $\sim 10\times$ the Higgs decay rate, however observations of the Higgs boson at the LHC indicate that the decay width to two photons is close to the standard model rate, see ATLAS Collaboration (2014). This is evidence against our current model, though any unified theory that introduces heavy charged fermions is going to have to deal with this issue. Note that in this analysis we have not included the loops of charged W's that can affect the $H \to 2\gamma$ rate.

2.6 Superfield expansions

To construct explicit superfield expansions we perform a series expansion in the property coordinates, multiplying each term by the corresponding space-time dependent field. We want the superfields to have no property overall, so we match up the terms in the series expansion with particle fields that have the opposite quantum numbers. For instance ζ^4 pairs with L^c not L. The last step is to include anti-particles and also the anti selfduality condition. We will list here the superfield expansion Ψ_N for the neutrinos:

$$\begin{aligned} 2\Psi_N =& \big(\zeta^{\bar{0}}N_1 + N_1^c\zeta^0\big)\big(1-\zeta^{1234\overline{1234}}\big) + \big(\zeta^{\bar{0}}N_2 + N_2^c\zeta^0\big)\big(\zeta^{4\bar{4}} + \zeta^{123\overline{123}}\big) \\ &+\big(\zeta^{\bar{0}}N_3 + N_3^c\zeta^0\big)\big(\zeta^{i\bar{i}} - \zeta^{0\bar{0}}\zeta^{j\bar{j}}\zeta^{k\bar{k}}/2\big)/\sqrt{3} + \big(\zeta^{\bar{0}}N_4 + N_4^c\zeta^0\big)\big(\zeta^{i\bar{i}}\zeta^{4\bar{4}} - \zeta^{j\bar{j}}\zeta^{k\bar{k}}/2\big) \end{aligned} \tag{2.13}$$

where the repeated indices are summed over 1, 2 and 3; for instance $\zeta^{i\bar{i}} = \zeta^{1\bar{1}} + \zeta^{2\bar{2}} + \zeta^{3\bar{3}}$. The idea behind this is that the property coordinates ζ^1, ζ^2 and ζ^3 correspond to the SU(3) colour group, so for the colourless parts of the superfield expansion to be invariant under transformations to the colour group we require the use of the SU(3) invariant $\zeta^{i\bar{i}}$. It is useful to note that Ψ_N satisfy $\int d^5\zeta d^5\bar{\zeta}(\bar{\Psi}_N\Psi_N) = \bar{N}_1N_1 + \bar{N}_2N_2 + \bar{N}_3N_3 + \bar{N}_4N_4$ as expected. The superfield expansions for the other fermions can be found in Delbourgo (2006b), and have similar properties.

Colourless neutral bosons (Higgs-like)

$$
\begin{array}{cc|cccccc}
 & & & & p= & & & \\
 & & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
 & 0 & \mathcal{M} & & & & \mathcal{H} & \\
 & 1 & & \mathcal{A},\mathcal{B},\mathcal{C} & & & & \times \\
 & 2 & * & & \mathcal{D},\mathcal{E},\mathcal{F},\mathcal{G} & & \times & \\
q= & 3 & & * & & \times & & \times \\
 & 4 & * & & * & & \times & \\
 & 5 & & * & & * & & \times
\end{array}
\tag{2.14}
$$

We can also perform a superfield expansion in colourless chargeless scalar bosons, which are Higgs-like particle fields. There are 9 such possible fields, which are shown in Table 2.14. All of these fields can act like the Higgs if they have non-zero expectation values. If we assume each of these fields takes on their expectation value then the full superfield expansion of the expectation value of the Higgs superfield Φ is:

$$
\begin{aligned}
2\langle\Phi\rangle =& \mathcal{M}(1-\zeta^{01234\overline{01234}}) + \mathcal{A}(\zeta^{0\bar{0}} + \zeta^{1234\overline{1234}}) \\
&+ \mathcal{B}(\zeta^{i\bar{i}} - \zeta^{04\overline{04}}\zeta^{j\bar{j}}\zeta^{k\bar{k}}/2)/\sqrt{3} + \mathcal{C}(\zeta^{4\bar{4}} - \zeta^{0123\overline{0123}}) \\
&+ \mathcal{D}(\zeta^{0\bar{0}}\zeta^{i\bar{i}} + \zeta^{4\bar{4}}\zeta^{j\bar{j}}\zeta^{k\bar{k}}/2)/\sqrt{3} + \mathcal{E}(\zeta^{4\bar{4}}\zeta^{i\bar{i}} + \zeta^{0\bar{0}}\zeta^{j\bar{j}}\zeta^{k\bar{k}}/2)/\sqrt{3} \\
&+ \mathcal{F}(\zeta^{i\bar{i}}\zeta^{j\bar{j}}/2 - \zeta^{04\overline{04}}\zeta^{k\bar{k}})/\sqrt{3} + \mathcal{G}(\zeta^{04\overline{04}} + \zeta^{123\overline{123}}) \\
&+ (\mathcal{H}\zeta^{1234} + \mathcal{H}^\star\zeta^{\overline{1234}})(1+\zeta^{0\bar{0}}).
\end{aligned}
\tag{2.15}
$$

Note that all the expectation values other than $\mathcal{H}$ are real. Checking this we see that $2\int d^5\zeta d^5\bar{\zeta}\,\langle\Phi\rangle^2 = \mathcal{M}^2 + \mathcal{A}^2 + \mathcal{B}^2 + \mathcal{C}^2 + \mathcal{D}^2 + \mathcal{E}^2 + \mathcal{F}^2 + \mathcal{G}^2 + 2\mathcal{H}^\star\mathcal{H}$. Like the standard model Higgs, our scalar fields can acquire non-zero expectation values via spontaneous symmetry breaking if we assume the following Lagrangian for the Higgs superfield:

$$
\mathcal{L} = \int d^5\zeta d^5\bar{\zeta}\Big((\partial\Phi)^2/2 + \mu^2\Phi^2/2 - \sqrt{2}f\Phi^3/3 - g\Phi^4/6\Big)
\tag{2.16}
$$

Note the cubic term which is included for generality, the standard spontaneous symmetry breaking Lagrangian does not use it. Using Mathematica we can expand out this Lagrangian fully and then take the variation with respect to each of the expectation values. Equating each of these to zero gives a set of algebraic conditions on the expectation values. The nine conditions are as follows:

$$
\begin{aligned}
0 &= \mathcal{M}\mu^2 - f\left(\mathcal{A}^2+\mathcal{B}^2+\mathcal{C}^2+\mathcal{D}^2+\mathcal{E}^2+\mathcal{F}^2+\mathcal{G}^2+2\mathcal{H}^2+\frac{3\mathcal{M}^2}{2}\right)\\
&- g\bigg(-\mathcal{A}\mathcal{B}\mathcal{D}-\mathcal{B}\mathcal{C}\mathcal{E}-\frac{2\mathcal{B}\mathcal{D}\mathcal{E}}{\sqrt{3}}-\frac{\mathcal{B}^2\mathcal{F}}{\sqrt{3}}-\mathcal{C}\mathcal{D}\mathcal{F}-\mathcal{A}\mathcal{E}\mathcal{F}-\mathcal{A}\mathcal{C}\mathcal{G}-\mathcal{B}\mathcal{F}\mathcal{G}-\mathcal{A}\mathcal{H}^2\\
&+\mathcal{A}^2\mathcal{M}+\mathcal{B}^2\mathcal{M}+\mathcal{C}^2\mathcal{M}+\mathcal{D}^2\mathcal{M}+\mathcal{E}^2\mathcal{M}+\mathcal{F}^2\mathcal{M}+\mathcal{G}^2\mathcal{M}+2\mathcal{H}^2\mathcal{M}+\frac{2\mathcal{M}^3}{3}\bigg).\\
0 &= \mathcal{A}\mu^2 - f\left(-\mathcal{B}\mathcal{D}-\mathcal{E}\mathcal{F}-\mathcal{C}\mathcal{G}-\mathcal{H}^2+2\mathcal{A}\mathcal{M}\right)\\
&- g\left(\frac{\mathcal{B}^2\mathcal{E}}{\sqrt{3}}+\mathcal{B}\mathcal{C}\mathcal{F}-\mathcal{B}\mathcal{D}\mathcal{M}-\mathcal{E}\mathcal{F}\mathcal{M}-\mathcal{C}\mathcal{G}\mathcal{M}-\mathcal{H}^2\mathcal{M}+\mathcal{A}\mathcal{M}^2\right)\\
0 &= \mathcal{B}\mu^2 - f\left(-\mathcal{A}\mathcal{D}-\mathcal{C}\mathcal{E}-\frac{2\mathcal{D}\mathcal{E}}{\sqrt{3}}-\frac{2\mathcal{B}\mathcal{F}}{\sqrt{3}}-\mathcal{F}\mathcal{G}+2\mathcal{B}\mathcal{M}\right)\\
&- g\left(\frac{2\mathcal{B}\mathcal{C}\mathcal{D}}{\sqrt{3}}+\frac{2\mathcal{A}\mathcal{B}\mathcal{E}}{\sqrt{3}}+\mathcal{A}\mathcal{C}\mathcal{F}+\frac{\mathcal{B}^2\mathcal{G}}{\sqrt{3}}-\mathcal{A}\mathcal{D}\mathcal{M}-\mathcal{C}\mathcal{E}\mathcal{M}-\frac{2\mathcal{D}\mathcal{E}\mathcal{M}}{\sqrt{3}}-\frac{2\mathcal{B}\mathcal{F}\mathcal{M}}{\sqrt{3}}-\mathcal{F}\mathcal{G}\mathcal{M}+\mathcal{B}\mathcal{M}^2\right)\\
0 &= \mathcal{C}\mu^2 - f(-\mathcal{B}\mathcal{E}-\mathcal{D}\mathcal{F}-\mathcal{A}\mathcal{G}+2\mathcal{C}\mathcal{M}) - g\left(\frac{\mathcal{B}^2\mathcal{D}}{\sqrt{3}}+\mathcal{A}\mathcal{B}\mathcal{F}-\mathcal{B}\mathcal{E}\mathcal{M}-\mathcal{D}\mathcal{F}\mathcal{M}-\mathcal{A}\mathcal{G}\mathcal{M}+\mathcal{C}\mathcal{M}^2\right)\\
0 &= \mathcal{D}\mu^2 - f\left(-\mathcal{A}\mathcal{B}-\frac{2\mathcal{B}\mathcal{E}}{\sqrt{3}}-\mathcal{C}\mathcal{F}+2\mathcal{D}\mathcal{M}\right) - g\left(\frac{\mathcal{B}^2\mathcal{C}}{\sqrt{3}}-\mathcal{A}\mathcal{B}\mathcal{M}-\frac{2\mathcal{B}\mathcal{E}\mathcal{M}}{\sqrt{3}}-\mathcal{C}\mathcal{F}\mathcal{M}+\mathcal{D}\mathcal{M}^2\right)\\
0 &= \mathcal{E}\mu^2 - f\left(-\mathcal{B}\mathcal{C}-\frac{2\mathcal{B}\mathcal{D}}{\sqrt{3}}-\mathcal{A}\mathcal{F}+2\mathcal{E}\mathcal{M}\right) - g\left(\frac{\mathcal{A}\mathcal{B}^2}{\sqrt{3}}-\mathcal{B}\mathcal{C}\mathcal{M}-\frac{2\mathcal{B}\mathcal{D}\mathcal{M}}{\sqrt{3}}-\mathcal{A}\mathcal{F}\mathcal{M}+\mathcal{E}\mathcal{M}^2\right)\\
0 &= \mathcal{F}\mu^2 - f\left(-\frac{\mathcal{B}^2}{\sqrt{3}}-\mathcal{C}\mathcal{D}-\mathcal{A}\mathcal{E}-\mathcal{B}\mathcal{G}+2\mathcal{F}\mathcal{M}\right)\\
&- g\left(\mathcal{A}\mathcal{B}\mathcal{C}-\frac{\mathcal{B}^2\mathcal{M}}{\sqrt{3}}-\mathcal{C}\mathcal{D}\mathcal{M}-\mathcal{A}\mathcal{E}\mathcal{M}-\mathcal{B}\mathcal{G}\mathcal{M}+\mathcal{F}\mathcal{M}^2\right)\\
0 &= \mathcal{G}\mu^2 - f(-\mathcal{A}\mathcal{C}-\mathcal{B}\mathcal{F}+2\mathcal{G}\mathcal{M}) - g\left(\frac{\mathcal{B}^3}{3\sqrt{3}}-\mathcal{A}\mathcal{C}\mathcal{M}-\mathcal{B}\mathcal{F}\mathcal{M}+\mathcal{G}\mathcal{M}^2\right)\\
0 &= \mathcal{H}\mu^2 - f(-\mathcal{A}\mathcal{H}+2\mathcal{H}\mathcal{M}) - g\left(-\mathcal{A}\mathcal{H}\mathcal{M}+\mathcal{H}\mathcal{M}^2\right)
\end{aligned}
\tag{2.17}
$$

The Mathematica code to derive these conditions is in the associated code for this thesis. Note that from now on we assume $\mathcal{H}$ is real to help make the conditions easier to work with. This system of equations can be used to calculate the expectation values of the 9 Higgs fields in our model. The benefit of doing all this is that we can have just one Higgs coupling, unlike the standard model where particle masses are essentially set by their independent coupling strengths to the Higgs field. In principle particle masses are derivable from a set of algebraic conditions, with only 3 free parameters from the Higgs superfield Lagrangian. In practice this system is quite hard to solve, especially when coupled with the fermionic mass matrices in an attempt to derive sensible particle masses. One extra complication to consider is that of not requiring the Higgs superfield to be anti selfdual, this would result in a doubling of the possible Higgs fields to 18. This would be a system of 18 simultaneous equations to solve, though each individually would be simpler.

2.7 Mass matrices

We can consider a Yukawa term describing the interaction between the fermion superfield and the Higgs superfield to get masses for fermions.

$$\mathcal{L}_{\text{yukawa}} = \int d^5\zeta d^5\bar{\zeta}\left(\bar{\Psi}\left\langle\Phi\right\rangle\Psi\right) \tag{2.18}$$

Expanding this out leads to a series of terms, which can be arranged into mass matrices. The elements of the mass matrix for a particle X are the coefficients of $\bar{X}_r X_s$ and $\bar{X^c}_r X^c_s$, for row r and column s. For example the Yukawa Lagrangian for neutrinos contains the following terms:

$$\mathcal{L}_{\text{yukawa}} \supset (\mathcal{A}/2+\mathcal{M})\bar{N}_1 N_1 + (\mathcal{A}/2+\mathcal{M})\bar{N^c}_1 N^c_1 + (\mathcal{C}/2-\mathcal{G}/2)\bar{N}_2 N_1 + (\mathcal{C}/2-\mathcal{G}/2)\bar{N^c}_2 N^c_1$$

This means the neutrino mass matrix has row 1, column 1 contain $(\mathcal{A}/2+\mathcal{M})+(\mathcal{A}/2+\mathcal{M}) = \mathcal{A}+2\mathcal{M}$ and row 2, column 1 contain $(\mathcal{C}/2-\mathcal{G}/2)+(\mathcal{C}/2-\mathcal{G}/2) = \mathcal{C}-\mathcal{G}$. The same pattern follows for the other terms and then other types of particle. The full set of mass matrices for the neutrinos, charged leptons, up and down quarks can be found over the page. The code used to derive them is available in the associated code with this thesis. A similar list can be found in Delbourgo (2006b), though there are a few small errors in that set as they were calculated by hand.

The mass matrices for the quarks and charged leptons are almost in a block diagonal structure, with the blocks linked by $\mathcal{H}$. These different blocks actually correspond to different lepton number, with the smaller block for down quarks and charged leptons having the incorrect number. These "different" states are still required as otherwise there would not be enough generations of up quark in our model. If the expectation value $\mathcal{H}$ is zero these blocks totally decouple from each other, indicating they could be considered a different type of particle. Rather than considering these to be exotic states, we need to keep in mind that this model does not include chirality yet, nor weak isospin quantum numbers. The lepton numbers given should only be thought of as a guide to be expanded on in the future.

The next step is to attempt to find a sensible set of masses for particles, we started by targeting the splitting between the neutrino masses and the top quark mass, which should be on the order of 10^{11} (≤ 1 eV for three lightest neutrinos, 173 GeV for the top quark). This was done by using the numerical equation solver in Mathematica to solve the system of conditions for the Higgs expectation values for a given f and g and then determining the eigenvalues of the mass matrices. The top quark mass would be the third highest eigenvalue of the up quark mass matrix, while the sum of the three smallest eigenvalues of the neutrino mass matrix needs to be on order $\sim$1 eV or smaller. We can then try this again with a different value of f and g as we search through the parameter space. We managed to achieve ratios of $\sim 10^7$ with these preliminary searches, while still short of the 10^{11} required this leaves opportunity for future work to re-examine this system and find a sensible set of parameters. This will hopefully lead to the discovery that the masses of fermions can be calculated from the 3 parameters in the Higgs Lagrangian. A more sophisticated take on solving the system will be required, for instance the numerical method used to solve the system of non-linear conditions on the expectation values did not pick out all possible solutions. It may also be possible to further analyse the eigenvalues of the mass matrices, through the quite nasty eigenvalue equations.

The mass matrices for the standard model fermions are as follows:

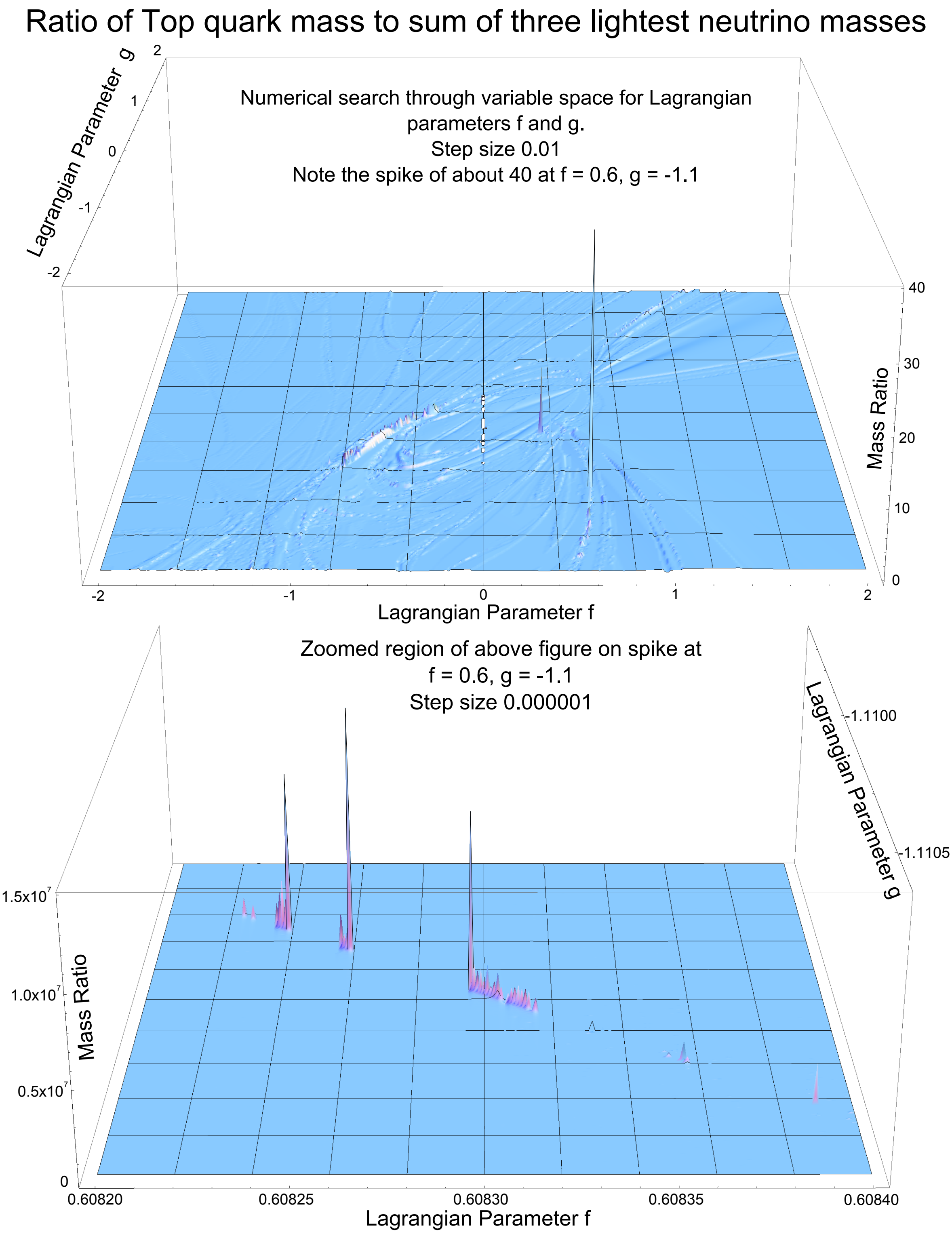
$$M(\text{Neutrino}) = \begin{pmatrix} \mathcal{A}+2\mathcal{M} & \mathcal{C}-\mathcal{G} & \mathcal{B}-\mathcal{D} & \mathcal{E}-\mathcal{F} \\ \mathcal{C}-\mathcal{G} & 2\mathcal{M} & \mathcal{F} & -\mathcal{B} \\ \mathcal{B}-\mathcal{D} & \mathcal{F} & \frac{2\mathcal{E}}{\sqrt{3}}+2\mathcal{M} & \frac{2\mathcal{B}}{\sqrt{3}}-\mathcal{C} \\ \mathcal{E}-\mathcal{F} & -\mathcal{B} & \frac{2\mathcal{B}}{\sqrt{3}}-\mathcal{C} & 2\mathcal{M} \end{pmatrix} \tag{2.19}$$

$$M(\text{Lepton}) = \begin{pmatrix} \mathcal{C}+2\mathcal{M} & -\mathcal{A}+\mathcal{G} & -\mathcal{B}+\mathcal{E} & \mathcal{D}-\mathcal{F} & -\mathcal{H} & -\mathcal{H} \\ -\mathcal{A}+\mathcal{G} & 2\mathcal{M} & \mathcal{F} & \mathcal{B} & -\mathcal{H} & 0 \\ -\mathcal{B}+\mathcal{E} & \mathcal{F} & \frac{2\mathcal{D}}{\sqrt{3}}+2\mathcal{M} & \mathcal{A}-\frac{2\mathcal{B}}{\sqrt{3}} & 0 & 0 \\ \mathcal{D}-\mathcal{F} & \mathcal{B} & \mathcal{A}-\frac{2\mathcal{B}}{\sqrt{3}} & 2\mathcal{M} & 0 & 0 \\ -\mathcal{H} & -\mathcal{H} & 0 & 0 & -\mathcal{G}+2\mathcal{M} & -\mathcal{A}-\mathcal{C} \\ -\mathcal{H} & 0 & 0 & 0 & -\mathcal{A}-\mathcal{C} & 2\mathcal{M} \end{pmatrix} \tag{2.20}$$

$$M(\text{Red up quarks}) = \begin{pmatrix} -\frac{\mathcal{E}}{\sqrt{3}}+2\mathcal{M} & -\frac{\mathcal{B}}{\sqrt{3}}-\mathcal{C} & -\mathcal{H} & 0 \\ -\frac{\mathcal{B}}{\sqrt{3}}-\mathcal{C} & 2\mathcal{M} & 0 & 0 \\ -\mathcal{H} & 0 & -\frac{\mathcal{F}}{\sqrt{3}}+2\mathcal{M} & -\frac{2\mathcal{B}}{\sqrt{3}} \\ 0 & 0 & -\frac{2\mathcal{B}}{\sqrt{3}} & 2\mathcal{M} \end{pmatrix} \tag{2.21}$$

$$M(\text{Red down quarks}) = \tag{2.22}$$

$$\begin{pmatrix} -\frac{\mathcal{B}}{\sqrt{3}}-2\mathcal{M} & \mathcal{A}-\frac{\mathcal{D}}{\sqrt{3}} & \frac{\mathcal{E}}{\sqrt{3}}-\mathcal{C} & \sqrt{\frac{2}{3}}(\mathcal{F}-\mathcal{B}) & \sqrt{\frac{2}{3}}(\mathcal{D}-\mathcal{E}) & \mathcal{G}-\frac{\mathcal{F}}{\sqrt{3}} & -\mathcal{H} & -\mathcal{H} \\ \mathcal{A}-\frac{\mathcal{D}}{\sqrt{3}} & -2\mathcal{M} & \frac{\mathcal{F}}{\sqrt{3}} & \sqrt{\frac{2}{3}}\mathcal{E} & \sqrt{\frac{2}{3}}\mathcal{B} & \mathcal{C} & -\mathcal{H} & 0 \\ \frac{\mathcal{E}}{\sqrt{3}}-\mathcal{C} & \frac{\mathcal{F}}{\sqrt{3}} & -2\mathcal{M} & -\sqrt{\frac{2}{3}}\mathcal{D} & \sqrt{\frac{2}{3}}\mathcal{B} & -\mathcal{A} & 0 & 0 \\ \sqrt{\frac{2}{3}}(\mathcal{F}-\mathcal{B}) & \sqrt{\frac{2}{3}}\mathcal{E} & -\sqrt{\frac{2}{3}}\mathcal{D} & -\mathcal{G}-2\mathcal{M} & -\mathcal{A}+\mathcal{C} & \sqrt{\frac{2}{3}}\mathcal{B} & 0 & 0 \\ \sqrt{\frac{2}{3}}(\mathcal{D}-\mathcal{E}) & \sqrt{\frac{2}{3}}\mathcal{B} & \sqrt{\frac{2}{3}}\mathcal{B} & -\mathcal{A}+\mathcal{C} & -2\mathcal{M} & 0 & 0 & 0 \\ \mathcal{G}-\frac{\mathcal{F}}{\sqrt{3}} & \mathcal{C} & -\mathcal{A} & \sqrt{\frac{2}{3}}\mathcal{B} & 0 & -2\mathcal{M} & 0 & 0 \\ -\mathcal{H} & -\mathcal{H} & 0 & 0 & 0 & 0 & \frac{\mathcal{D}}{\sqrt{3}}-2\mathcal{M} & \mathcal{A}+\frac{\mathcal{B}}{\sqrt{3}} \\ -\mathcal{H} & 0 & 0 & 0 & 0 & 0 & \mathcal{A}+\frac{\mathcal{B}}{\sqrt{3}} & -2\mathcal{M} \end{pmatrix}$$



Results of a numerical search through the parameter space of f and g from the Bosonic lagrangian in Equation 2.16 as described in Section 2.7. Note the spikes of up to $\sim 10^7$ when the resolution was increased sufficiently. It is quite likely further numerical analyses would be able to reach the ratios of 10^{11} required, the next step would be to place further restrictions to try and derive the standard model particle masses.

2.8 Conclusions

In this chapter we have described a grand unified model, through the introduction of property coordinates to space-time. Five such coordinates are required to produce all known observed particles, though this does introduce several exotic particles as well. The number of states is cut down by the use of an anti selfduality condition, but even with this the extra charged particles introduced should in theory increase the decay rate of the Higgs Boson into two photons by a factor of $\sim 10\times$. This is not matched by experiments performed at the LHC, which indicate the Higgs decays at the standard model rate. This will be a significant challenge for most grand unified theories however, and so should not be seen as too discouraging. We can perform explicit superfield expansions in the property coordinates to get fermionic and bosonic superfields. We can also construct a Higgs like superfield, with 9 possible Higgs like particles. These Higgs fields can obtain non-zero expectation values through spontaneous symmetry breaking, and then via a Yukawa interaction with fermion fields produce particle masses. We attempted to analyse the system of mass matrices produced and got some promising results, with indication that the neutrino and up quark masses can be separated. This leaves an opportunity for future work to explore this and also to perhaps consider other mechanisms like renormalisation to produce the observed particle mass spectrum.

Chapter 3

General Relativity on a graded manifold

The most studied types of supergravity theories are those involving additional commutative coordinates like Kaluza-Klein or the spinorial coordinates introduced by supersymmetry. Manifolds with just an even/odd grading structure, like those produced by including property coordinates, have been studied far less. Both Arnowitt and Nath (1976) and Asorey and Lavrov (2009) consider the properties of such graded manifolds but make use of right handed derivatives. With a graded manifold the side a derivative acts from becomes important, we wish to only use left handed derivatives in this work as they are more familiar. This also allows us to use tensor comma notation unambiguously. As a result of this the starting point for this chapter is Delbourgo (2006a) which uses left handed derivatives exclusively.

3.1 Notation

The addition of extra anticommutative coordinates to space time results in a graded manifold, where the standard space time is even and the property sector is odd. The notation used in this thesis will be to define uppercase roman indices (M, N, L, etc) to run over all the dimensions of space, time and property and hence have mixed grading. Lower case roman indices (m, n, l, etc) will correspond to even graded space time, and Greek characters (μ, ν, λ, etc) will correspond to the odd graded property sector. As the grading of the general uppercase roman indices isn't specified, expressions involving those indices may depend on the grading of the index. For example for a given space-time-property coordinate $X^M = (X^m, X^\mu)$:

$$X^m X^N = X^N X^m, \tag{3.1}$$

$$X^\mu X^\nu = -X^\nu X^\mu. \tag{3.2}$$

The coordinates commute if one of them is graded even, and anti-commute if both are graded odd. This can be expressed as follows:

$$X^M X^N = (-1)^{\text{Grading}[M]\text{Grading}[N]} X^N X^M \tag{3.3}$$

where Grading[M] = 0 if M is even graded (i.e. M = m) or Grading[M] = 1 if M is odd graded (i.e. M = μ). Now as a shorthand because these type of expressions will show up a lot, the Grading function will not be written out explicitly. Whenever an expression contains uppercase roman indices in the exponent of (-1) these will be assumed to be the gradings of those indices. Hence the above expression can be written as:

$$X^M X^N = (-1)^{MN} X^N X^M. \tag{3.4}$$

We call this factor of $(-1)^{MN}$ a sign factor. Now for comparison Asorey and Lavrov (2009) use the notation $(-1)^{\varepsilon_m \varepsilon_n}$ and Delbourgo (2006a) uses the notation $(-1)^{[M][N]}$, we will use $(-1)^{MN}$ in this work as the meaning is still clear and there is less clutter involved.

Note that Einstein summation applies in this thesis, but not to gradings present in the exponent of (-1). For example $(-1)^{NM} X^N X_M$ involves no summation, but $(-1)^N X_N X^N$ $= X_n X^n - X_\nu X^\nu$ has summation over the X's present, with the grading of N just carried along to give the correct sign.

3.2 Vectors

General vectors have the same commutation properties as the corresponding space-time-property coordinates, i.e:

$$V^M V^N = (-1)^{MN} V^N V^M. \tag{3.5}$$

This also extends to covariant vectors and products between covariant and contravariant vectors.

$$V^M V_N = (-1)^{MN} V_N V^M \tag{3.6}$$

Any permutation of a product of vectors can be considered in the same way.

$$V^M V^N V^L V^P = (-1)^{P(L+N+M)} V^P V^M V^N V^L = (-1)^{P(L+N+M)+M(N+L)} V^P V^N V^L V^M \tag{3.7}$$

Raising and lowering of vector indices via the metric is the same as standard GR, except an order has to be specified due to non-commutation. The convention we adopt is to always have an "up" index followed by a summed "down" index. For example:

$$V^M G_{MN} = V_N, \quad G^{NM} V_M = V^N, \tag{3.8}$$

where Einstein summation is performed as usual. In some cases it will not be possible to have sums with an up index followed immediately by its down index, in these cases a sign factor has to be introduced that effectively shuffles the indices so that an up then down summation occurs. This rule will be discussed more in Section 3.5.

3.3 Differentiation

The order expressions appear in becomes very important when dealing with differentiation of a manifold with graded symmetry. The standard convention is for differentiation to act on the left, i.e: $f_{,M} = \frac{\partial f}{\partial X^M} = \frac{\partial}{\partial X^M} f$ and so that is what we will use here. The chain rule is then given by

$$\frac{\partial f}{\partial X'^N} = \frac{\partial X^L}{\partial X'^N}\frac{\partial f}{\partial X^L} = \frac{\partial}{\partial X'^N}\left(X^L\right)\frac{\partial}{\partial X^L} f, \tag{3.9}$$

with differentials written as

$$df = dX^L \frac{\partial f}{\partial X^L}, \tag{3.10}$$

in that particular order. Note that if the order of the terms is to be swapped then a sign factor must be introduced based on the grading of the terms present, i.e:

$$df = dX^L \frac{\partial f}{\partial X^L} = (-1)^{(\mathrm{Grading}[f]+L)L}\frac{\partial f}{\partial X^L} dX^L = (-1)^{\mathrm{Grading}[f]L+L}\frac{\partial f}{\partial X^L} dX^L, \tag{3.11}$$

$$\frac{\partial X^L}{\partial X'^N}\frac{\partial f}{\partial X^L} = (-1)^{(\mathrm{Grading}[f]+L)(L+N)}\frac{\partial f}{\partial X^L}\frac{\partial X^L}{\partial X'^N}. \tag{3.12}$$

In terms of differentials the product rule is done as usual:

$$d(fgh) = df\ g\ h + f\ dg\ h + f\ g\ dh. \tag{3.13}$$

However when differentiation with respect to a variable is introduced, sign factors have to be included as the differential operator is permuted through the expression:

$$\frac{\partial (fgh)}{\partial X^M} = \frac{\partial f}{\partial X^M} gh + (-1)^{\mathrm{Grading}[f]M} f \frac{\partial g}{\partial X^M} h + (-1)^{M(\mathrm{Grading}[f]+\mathrm{Grading}[g])} fg \frac{\partial h}{\partial X^M}. \tag{3.14}$$

As the gradings can only be 0 or 1, and $(-1)^2 = 1 = (-1)^0$, this means the gradings act as the additive group of integers modulo 2. This allows for simplification of sign factors, for example $(-1)^{N+N} = 1$.

3.4 Contravariant and Covariant Vectors

The chain rule gives how dX^M transforms, $dX'^M = dX^N \frac{\partial X'^M}{\partial X^N}$. If we require that a contravariant vector V^M transforms like dX^M we get

$$V'^M = V^N \frac{\partial X'^M}{\partial X^N} \text{ and similarly } V^M = V'^N \frac{\partial X^M}{\partial X'^N} \tag{3.15}$$

We then expect that $V^M V_M$ will be invariant:

$$V'^N V'_N = V^M V_M = V'^N \frac{\partial X^M}{\partial X'^N} V_M \tag{3.16}$$

$$\therefore V'_N = \frac{\partial X^M}{\partial X'^N} V_M \text{ and similarly } V_N = \frac{\partial X'^M}{\partial X^N} V'_M \tag{3.17}$$

This is the rule for covariant vectors. These can be used to find the transformation laws for any tensor, by requiring that it transforms like a product of contravariant and covariant vectors. For example a rank 2 covariant tensor, G_{MN} should transform like $V_M V_N$.

$$V'_M V'_N = \frac{\partial X^S}{\partial X'^M} V_S \frac{\partial X^R}{\partial X'^N} V_R = (-1)^{S(R+N)} \frac{\partial X^S}{\partial X'^M} \frac{\partial X^R}{\partial X'^N} V_S V_R \tag{3.18}$$

$$\therefore G'_{MN} = (-1)^{S(R+N)} \frac{\partial X^S}{\partial X'^M} \frac{\partial X^R}{\partial X'^N} G_{SR} \tag{3.19}$$

To simplify expressions like the above, we will introduce shorthand notation:

$$\frac{\partial X^M}{\partial X'^N} = \partial_{\prime N}{}^M. \tag{3.20}$$

The above then become:

$$V'^M = V^N \partial_N{}'^M, \tag{3.21}$$

$$V'_M = \partial_{\prime M}{}^N V_N, \tag{3.22}$$

$$G'_{MN} = (-1)^{R(S+N)} \partial_{\prime M}{}^R \partial_{\prime N}{}^S G_{RS}, \tag{3.23}$$

also similarly

$$G'^{MN} = (-1)^{S(R+M)} G^{RS} \partial_R{}'^M \partial_S{}'^N. \tag{3.24}$$

3.5 Up then down summation rule

As can be seen from Equation 3.24 if the indices are not directly arranged in the up then down summation order a sign factor must be introduced to ensure the correct transformation properties of the summed expression. This sign factor is equivalent to permuting the indices in the expression through each other until the correct order is achieved. Delbourgo (2006a) did not strictly enforce this up then down summation rule, a significant amount of time was spent testing the formalism from that paper. It was found there were inconsistencies and contradictions that could be produced by considering various contractions of the metric with vectors. For instance $V^M G_{MN}$ must transform in the same way V_N does. Adopting the up then down summation rule fixed these problems and eliminated all the inconsistencies that were present before.

A simple example to demonstrate the importance of the up then down summation rule goes as follows, consider $(-1)^N G^{MN} V_N$, expanding this out we get:

$$(-1)^N G^{MN} V_N = G^{Mn} V_n - G^{M\nu} V_\nu. \tag{3.25}$$

This cannot be simplified to be in terms of V^M due to the minus sign present. If the metric is block diagonal then this simplification could be performed, but in general it is not possible. The presence of a summed index's grading in the sign factor of an expression has to be treated carefully, the up then down rule takes care of this. Rearranging the above expression we get the following:

$$(-1)^N G^{MN} V_N = V_N G^{NM}. \tag{3.26}$$

We can see here that the summation is in fact down then up, the factor of $(-1)^N$ was what broke the up then down rule. Note that Equation 3.26 used the symmetry of the metric, which will be discussed in Section 3.7.

The resulting rule is as follows: Tensor summations must be performed with a raised "up" index followed by a lower "down" index. In the case where this isn't possible, due to for instance an index being part of a tensor, then a sign factor must be introduced that is equivalent to a permutation of the indices to produce the up then down summation order. The up then down rule is used as part of the rules to test that tensor equations are valid. The process to check for validity of a tensor equation is as follows: first check the free indices on both sides match, secondly check that the up then down rule is obeyed in sums, finally check that the free tensor indices appear in the same order on both sides after taking into account permutations by the sign factor. This check on tensor equations can in fact be used to get the form of several identities that are painstakingly derived by hand later in this chapter.

3.6 The singlet

The singlet for a type (1,1) tensor is $\delta_M{}^L$. Note that there is an ordering to the indices, which is normally not of importance in standard GR but is here. First we will check that it is indeed invariant under coordinate transformations, we expect it to transform like $V_M V^L$.

$$V'_M V'^L = \partial_{\prime M}{}^R V_R V^S \partial_S{}'^L \tag{3.27}$$

Thus we expect

$$\delta'_M{}^L = \partial_{\prime M}{}^R \delta_R{}^S \partial_S{}'^L. \tag{3.28}$$

The summation over S and R can be performed to give

$$\delta'_M{}^L = \partial_{\prime M}{}^R \partial_R{}'^L = \frac{\partial X^R}{\partial X'^M}\frac{\partial X'^L}{\partial X^R} = \frac{\partial X'^L}{\partial X'^M} = \delta_M{}^L. \tag{3.29}$$

Thus $\delta_M{}^L$ transforms into itself and is hence a singlet. Let us now consider $\delta^L{}_M$ and see how it transforms.

$$V'^L V'_M = V^S \partial_S{}'^L \partial_{\prime M}{}^R V_R = (-1)^{S(S+L)+R(R+M)} \partial_S{}'^L V^S V_R \partial_{\prime M}{}^R \tag{3.30}$$

Thus we expect:

$$\delta'^L{}_M = (-1)^{S(S+L)+R(R+M)} \partial_S{}'^L \delta^S{}_R \partial_{\prime M}{}^R. \tag{3.31}$$

This becomes a little easier to follow if we introduce the following symmetry property:

$$\delta^M{}_N = (-1)^{MN} \delta_N{}^M. \tag{3.32}$$

The expression then becomes:

$$\delta'^L{}_M = (-1)^{R+RL+RM} \partial_R{}'^L \partial_{\prime M}{}^R = (-1)^{LM} \partial_{\prime M}{}^R \partial_R{}'^L = (-1)^{LM} \delta_M{}^L = \delta^L{}_M. \tag{3.33}$$

Thus, $\delta^L{}_M$ is also an invariant. In this work we will stick to using the indices in the order of $\delta_M{}^L$, as it usually makes simplification easier.

3.7 The metric tensor

The metric tensor G_{MN} is used in a similar manner to the metric from standard GR. By convention we make the metric graded symmetric, $G_{MN} = (-1)^{MN} G_{NM}$, which is consistent with standard GR for the even space-time case. We then want to have an inverse metric G^{NL} that when multiplied by the metric produces the singlet $\delta_M{}^L$. The two identities that result from obeying the rules for tensor equations are as follows:

$$G^{LN} G_{NM} = (-1)^{ML} \delta_M{}^L, \tag{3.34}$$

$$(-1)^N G_{MN} G^{NL} = \delta_M{}^L. \tag{3.35}$$

The sign factor in the first is required to have M and L appear in the correct order on both sides, the factor of $(-1)^N$ in the second identity is necessary due to the N sum not being up then down. From this the symmetry properties of the inverse metric can be determined.

$$\begin{aligned} G^{MN} G_{NL} &= (-1)^{ML} \delta_L{}^M = (-1)^{N+ML} G_{LN} G^{NM} \\ &= (-1)^{MN+NL} G^{NM} G_{LN} = (-1)^{MN} G^{NM} G_{NL} \end{aligned} \tag{3.36}$$

This is consistent with the inverse metric satisfying $G^{MN} = (-1)^{MN} G^{NM}$, i.e. graded symmetric like the metric tensor.

The metric tensor G_{MN} and its inverse G^{MN} are used to raise and lower indices similar to standard GR. Care has to be taken with the order the metric is applied in though, a sign factor must be included if the summation is not up then down.

$$\begin{aligned} V^M &= G^{MN} V_N = (-1)^{N+MN} V_N G^{MN} = (-1)^N V_N G^{NM} \\ V_M &= V^N G_{NM} = (-1)^{N+MN} G_{NM} V^N = (-1)^N G_{MN} V^N \\ T^M{}_L &= G^{MN} T_{NL} = (-1)^{(M+N)(N+L)} T_{NL} G^{MN} = (-1)^{L(M+N)+N} T_{NL} G^{NM} \end{aligned} \tag{3.37}$$

For higher rank tensors, both the metric's indices must be permuted through the tensor. The summed one is brought through to ensure that the summation is up then down, the other to ensure the new tensor transforms correctly with that index in the right spot.

$$T_M{}^R{}_L = (-1)^{(R+N)M} G^{RN} T_{MNL} \tag{3.38}$$

3.8 Covariant Derivative of Covariant Vector

The standard connection coefficients in the case of zero torsion are defined to be:

$$\Gamma_{mn}{}^p = \frac{1}{2} \left(g_{lm,n} + g_{ln,m} - g_{mn,l} \right) g^{lp}. \tag{3.39}$$

We take this as the starting point for our covariant derivative, extending it to a graded manifold by including sign factors. Note that we are implicitly assuming zero torsion, this is done to simplify the calculations which are difficult enough already. Future work could consider what effect torsion would have.

$$\Gamma_{MN}{}^P = \frac{1}{2} \Big((-1)^{X_{LMN}} G_{LM,N} + (-1)^{Y_{LNM}} G_{LN,M} - (-1)^{Z_{MNL}} G_{MN,L} \Big) G^{LP} \tag{3.40}$$

The unknowns that need to be determined are X_{LMN} which depends on L, M and N, Y_{LNM} which depends on L, N and M and Z_{MNL} which depends on M, N and L. This is done by looking at the covariant derivative of a covariant vector and ensuring it transforms correctly as a rank 2 covariant tensor. We take the covariant derivative of a covariant vector to be:

$$A_{M;N} = (-1)^{W_{MN}} A_{M,N} - \Gamma_{MN}{}^P A_P, \tag{3.41}$$

where again W_{MN} is an unknown sign factor. Expanding this out and finding the conditions on the sign factors so that all second derivatives cancel and the remaining terms transform as a rank 2 covariant tensor gives the following:

$$A_{M;N} = (-1)^{MN} A_{M,N} - \Gamma_{MN}{}^P A_P, \tag{3.42}$$

where

$$\Gamma_{MN}{}^{P} = \frac{1}{2}\Big((-1)^{MN+LM+L}G_{LM,N} + (-1)^{LN+L}G_{LN,M} - (-1)^{LM+LN+L}G_{MN,L}\Big)G^{LP}. \tag{3.43}$$

As the initial starting point was for a zero torsion tensor, we would predict that our connection coefficient to be graded symmetric on interchange of the lower two indices. As expected we find:

$$\Gamma_{MN}{}^{P} = (-1)^{MN}\Gamma_{NM}{}^{P}. \tag{3.44}$$

Now define the covariant differentiation operator ∇ to act as follows:

$$\begin{aligned}\nabla_N A_M &= (-1)^{MN}A_{M;N}\\ &= A_{M,N} - (-1)^{MN}\Gamma_{MN}{}^{L}A_L\\ &= A_{M,N} - \Gamma_{NM}{}^{L}A_L.\end{aligned} \tag{3.45}$$

Doing this means the ∇_N operator acts on the left like a partial derivative, but when written with semicolon notation like in $A_{M;N}$ the N can act on the right correctly.

3.9 Covariant Differentiation of a Contravariant vector

The covariant derivative of a contravariant vector can be found in a similar manner as above, or by considering the covariant derivative of a scalar. For instance take:

$$\big(A^M A_M\big)_{;N} = \big(A^M A_M\big)_{,N} = A^M{}_{,N}A_M + (-1)^{MN}A^M A_{M,N}. \tag{3.46}$$

Expanding out the covariant derivative however gives the following:

$$\begin{aligned}\big(A^M A_M\big)_{;N} =&(-1)^{MN}A^M{}_{;N}A_M + A^M A_{M;N}\\ =&(-1)^{MN}A^M{}_{;N}A_M + A^M\big[(-1)^{MN}A_{M,N} - \Gamma_{MN}{}^{P}A_P\big].\end{aligned} \tag{3.47}$$

Equating both of these and rearranging we get:

$$A^M{}_{;N}A_M =(-1)^{MN}A^M{}_{,N}A_M + (-1)^{MN}A^P\Gamma_{PN}{}^{M}A_M. \tag{3.48}$$

Which is satisfied by

$$A^M{}_{;N} =(-1)^{MN}A^M{}_{,N} + (-1)^{MN}A^P\Gamma_{PN}{}^{M}. \tag{3.49}$$

3.10 Covariant Differentiation of a Covariant Tensor

Consider a rank 2 covariant tensor T_{MN}. We want this to transform like the product of two covariant vectors.

$$T_{MN} = V_M W_N \tag{3.50}$$

The covariant derivative of this tensor is then given by:

$$\begin{aligned} T_{MN;L} &= (-1)^{L(M+N)} \nabla_L T_{MN} \\ &= (-1)^{L(M+N)} \nabla_L (V_M W_N). \end{aligned} \tag{3.51}$$

The product rule can now be applied to this expression, taking care to permute the covariant differentiation operator through the vectors.

$$\begin{aligned} T_{MN;L} &= (-1)^{L(M+N)} \nabla_L (V_M W_N) \\ &= (-1)^{L(M+N)} \left[(\nabla_L V_M) W_N + (-1)^{LM} V_M (\nabla_L W_N)\right] \\ &= (-1)^{L(M+N)} \left[(V_{M,L} - \Gamma_{LM}{}^K V_K) W_N + (-1)^{LM} V_M (W_{N,L} - \Gamma_{LN}{}^K W_K)\right] \\ &= (-1)^{L(M+N)} \left[V_{M,L} W_N + (-1)^{LM} V_M W_{N,L} - \Gamma_{LM}{}^K V_K W_N - (-1)^{LM} V_M \Gamma_{LN}{}^K W_K\right] \\ &= (-1)^{L(M+N)} \left[T_{MN,L} - \Gamma_{LM}{}^K T_{KN} - (-1)^{LM+M(K+L+N)} \Gamma_{LN}{}^K T_{MK}\right] \\ &= (-1)^{L(M+N)} \left[T_{MN,L} - \Gamma_{LM}{}^K T_{KN} - (-1)^{M(K+N)} \Gamma_{LN}{}^K T_{MK}\right] \end{aligned} \tag{3.52}$$

This process can also be extended to higher rank covariant tensors. For example the covariant derivative of the covariant Riemann curvature tensor ends up as follows:

$$\begin{aligned} R_{JKLM;N} =& (-1)^{N(J+K+L+M)} \Big[R_{JKLM,N} - \Gamma_{NJ}{}^R R_{RKLM} - (-1)^{J(R+K)} \Gamma_{NK}{}^R R_{JRLM} \\ & - (-1)^{(J+K)(R+L)} \Gamma_{NL}{}^R R_{JKRM} - (-1)^{(J+K+L)(R+M)} \Gamma_{NM}{}^R R_{JKLR} \Big]. \end{aligned} \tag{3.53}$$

As a test of the formalism the covariant derivative of the metric can be considered, which should come out to zero.

$$\begin{aligned} G_{MN;L} =& (-1)^{L(M+N)} \left[G_{MN,L} - \Gamma_{LM}{}^K G_{KN} - (-1)^{M(K+N)} \Gamma_{LN}{}^K G_{MK}\right] \\ G_{MN;L} =& (-1)^{L(M+N)} \Big[G_{MN,L} \\ & - \frac{1}{2}\left((-1)^{LM+SL+S} G_{SL,M} + (-1)^{SM+S} G_{SM,L} - (-1)^{SL+SM+S} G_{LM,S}\right) G^{SK} G_{KN} \\ & - (-1)^{M(K+N)} \frac{1}{2}\left((-1)^{LN+SL+S} G_{SL,N} + (-1)^{SN+S} G_{SN,L} - (-1)^{SL+SN+S} G_{LN,S}\right) \\ & G^{SK} G_{MK} \Big] \end{aligned} \tag{3.54}$$

After summing over K and S we get the following:

$$G_{MN;L} = (-1)^{L(M+N)}\Big[G_{MN,L} - \frac{1}{2}\Big((-1)^{ML+NL}G_{NL,M} + G_{MN,L} - (-1)^{NL+ML+MN}G_{ML,N} + (-1)^{NL+ML+MN}G_{ML,N} + G_{MN,L} - (-1)^{ML+NL}G_{NL,M}\Big)\Big] \quad (3.55)$$

which is zero as expected.

3.11 Riemann curvature tensor

We now move to construct the Riemann curvature tensor. We will begin by considering the following expression:

$$A_{K;L;M} - (-1)^{LM}A_{K;M;L} = \Big[(-1)^{KL}\Gamma_{KM}{}^{S}{}_{,L} - (-1)^{KM+LM}\Gamma_{KL}{}^{S}{}_{,M} + (-1)^{LM}\Gamma_{KM}{}^{R}\Gamma_{RL}{}^{S} - \Gamma_{KL}{}^{R}\Gamma_{RM}{}^{S}\Big]A_S. \quad (3.56)$$

We reached this by expanding out the covariant derivative of $A_{K;L}$ as a rank two tensor and then expanding out the covariant derivative of A_K. There were partial derivatives of A_K present, however they all canceled as expected. Now let us define the curvature tensor:

$$(-1)^{S(K+L+M)}R^{S}{}_{KLM}A_S = A_{K;L;M} - (-1)^{LM}A_{K;M;L}. \quad (3.57)$$

As Equation 3.57 has to be true for all A_S comparing this with Equation 3.56 results in:

$$R^{S}{}_{KLM} = (-1)^{S(K+L+M)}\Big[(-1)^{KL}\Gamma_{KM}{}^{S}{}_{,L} - (-1)^{KM+LM}\Gamma_{KL}{}^{S}{}_{,M} + (-1)^{LM}\Gamma_{KM}{}^{R}\Gamma_{RL}{}^{S} - \Gamma_{KL}{}^{R}\Gamma_{RM}{}^{S}\Big]. \quad (3.58)$$

The fully covariant curvature tensor is given by

$$R_{JKLM} = (-1)^{(S+J)(K+L+M)}R^{S}{}_{KLM}G_{SJ}, \quad (3.59)$$

$$\therefore R_{JKLM} = (-1)^{J(K+L+M)}\Big[(-1)^{KL}\Gamma_{KM}{}^{S}{}_{,L} - (-1)^{KM+LM}\Gamma_{KL}{}^{S}{}_{,M} + (-1)^{LM}\Gamma_{KM}{}^{R}\Gamma_{RL}{}^{S} - \Gamma_{KL}{}^{R}\Gamma_{RM}{}^{S}\Big]G_{SJ}. \quad (3.60)$$

Now we want to check the symmetry properties of this to ensure it is correct. $L \leftrightarrow M$ symmetry on the last two indices is quite straight foward to check:

$$\begin{aligned}
&R_{JKML} = \\
&(-1)^{J(K+L+M)}\Big[(-1)^{KM}\Gamma_{KL}{}^{S}{}_{,M} - (-1)^{KL+LM}\Gamma_{KM}{}^{S}{}_{,L} + (-1)^{LM}\Gamma_{KL}{}^{R}\Gamma_{RM}{}^{S} - \Gamma_{KM}{}^{R}\Gamma_{RL}{}^{S}\Big]G_{SJ} \\
&= -(-1)^{LM}R_{JKLM}.
\end{aligned} \tag{3.61}$$

Thus the fully covariant Riemann curvature tensor is graded antisymmetric on the last two indices as required. Testing the other symmetry properties is significantly harder to do so we will break the problem into parts. The first step is to expand out the covariant Riemann curvature tensor in terms of derivatives of the metric as follows:

$$R_{JKLM} = R(flat)_{JKLM} + C_{JKLM} - (-1)^{LM}C_{JKML}, \tag{3.62}$$

where

$$\begin{aligned}
R(flat)_{JKLM} =& \frac{1}{2}\Big[(-1)^{JK+JL}G_{JM,LK} - (-1)^{KL}G_{KM,LJ} - (-1)^{LM+JK+JM}G_{JL,MK} \\
&+ (-1)^{KM+LM}G_{KL,MJ}\Big],
\end{aligned} \tag{3.63}$$

$$\begin{aligned}
C_{JKLM} =& \frac{1}{4}(-1)^{J(K+L+M)+A+LM}\Big[(-1)^{KM}G_{KA,M} + G_{MA,K} - (-1)^{AK+AM}G_{KM,A}\Big]G^{AS} \\
&\Big[-(-1)^{LS}G_{SJ,L} + G_{LJ,S} - (-1)^{JS+JL}G_{SL,J}\Big].
\end{aligned} \tag{3.64}$$

$R(flat)_{JKLM}$ is the curvature tensor in a local frame, where first derivatives of the metric vanish. The C_{JKLM} terms form the correction due to general curved space.

3.12 Symmetry of Riemann curvature tensor in a local frame

Let us now examine the symmetry properties of the curvature tensor in a local frame. We are left with:

$$\begin{aligned}
R_{JKLM} =& R(flat)_{JKLM} \\
=& \frac{1}{2}\Big[(-1)^{JK+JL}G_{JM,LK} - (-1)^{KL}G_{KM,LJ} \\
&- (-1)^{LM+JK+JM}G_{JL,MK} + (-1)^{KM+LM}G_{KL,MJ}\Big].
\end{aligned} \tag{3.65}$$

Start with $J \leftrightarrow K$ symmetry:

$$\begin{aligned}
R(flat)_{KJLM} =& \frac{1}{2}\Big[(-1)^{KJ+KL}G_{KM,LJ} - (-1)^{JL}G_{JM,LK} \\
& - (-1)^{LM+KJ+KM}G_{KL,MJ} + (-1)^{JM+LM}G_{JL,MK}\Big] \\
=& -(-1)^{JK}\frac{1}{2}\Big[(-1)^{JL+JK}G_{JM,LK} - (-1)^{KL}G_{KM,LJ} \\
& - (-1)^{JM+LM+JK}G_{JL,MK} + (-1)^{LM+KM}G_{KL,MJ}\Big] \\
=& -(-1)^{JK}R(flat)_{JKLM}.
\end{aligned} \tag{3.66}$$

$M \leftrightarrow L$ symmetry has already been done, now to check $(J+K) \leftrightarrow (L+M)$ symmetry:

$$\begin{aligned}
R(flat)_{LMJK} =& \frac{1}{2}\Big[(-1)^{LM+LJ}G_{LK,JM} - (-1)^{MJ}G_{MK,JL} \\
& - (-1)^{JK+LM+LK}G_{LJ,KM} + (-1)^{MK+JK}G_{MJ,KL}\Big] \\
=& \frac{1}{2}(-1)^{(J+K)(M+L)}\Big[(-1)^{JK+JL}G_{JM,LK} - (-1)^{KL}G_{KM,LJ} \\
& - (-1)^{JK+LM+JM}G_{JL,MK} + (-1)^{LM+KM}G_{KL,MJ}\Big] \\
=& (-1)^{(J+K)(M+L)}R(flat)_{JKLM}.
\end{aligned} \tag{3.67}$$

Thus the curvature tensor has the expected symmetry properties in a local frame.

3.13 General symmetry of Riemann curvature tensor

Now we will look at the symmetry properties of the other terms in the curvature tensor. We will need some identities involving the C_{JKLM} first. The following can all be derived by expanding and swapping around terms then making use of the symmetry of the metric:

$$-(-1)^{LM}C_{JKML} = -(-1)^{JK}C_{KJLM}, \tag{3.68}$$

$$C_{LMJK} = (-1)^{(J+K)(L+M)}C_{JKLM}. \tag{3.69}$$

Now from before the curvature tensor can be written as:

$$R_{JKLM} = R(flat)_{JKLM} + C_{JKLM} - (-1)^{LM}C_{JKML}. \tag{3.70}$$

To test $(J \leftrightarrow K)$ symmetry we can use Equation 3.68 to write:

$$R_{JKLM} = R(flat)_{JKLM} + C_{JKLM} - (-1)^{JK}C_{KJLM}. \tag{3.71}$$

Making use of symmetry of the $R(flat)$ term from Equation 3.66 we get:

$$\begin{aligned} R_{JKLM} =& -(-1)^{JK}\left[R(flat)_{KJLM} + C_{KJLM} - (-1)^{JK}C_{JKLM}\right] \\ =& -(-1)^{JK}R_{KJLM}. \end{aligned} \tag{3.72}$$

To test $(J+K) \leftrightarrow (L+M)$ symmetry we follow a similar procedure. Using Equation 3.69 on Equation 3.70, and then making use of the symmetry of $R(flat)$ from Equation 3.67 we can show:

$$\begin{aligned} R_{JKLM} =&(-1)^{(J+K)(L+M)}\left[R_{LMJK} + C_{LMJK} - (-1)^{LM}C_{MLJK}\right] \\ =&(-1)^{(J+K)(L+M)}R_{LMJK}. \end{aligned} \tag{3.73}$$

Hence we have shown the Riemann curvature tensor obeys the following symmetry properties in general:

$$R_{JKLM} = -(-1)^{JK}R_{KJLM}, \tag{3.74}$$

$$R_{JKLM} = -(-1)^{LM}R_{JKML}, \tag{3.75}$$

$$R_{JKLM} =(-1)^{(J+K)(L+M)}R_{LMJK}. \tag{3.76}$$

3.14 First Bianchi identity

Let us look for a cyclic identity of the following form:

$$(-1)^{X_{JKLM}}R_{JKLM} + (-1)^{Y_{JMKL}}R_{JMKL} + (-1)^{Z_{JLMK}}R_{JLMK} = 0, \tag{3.77}$$

where X_{JKLM}, Y_{JMKL} and Z_{JLMK} are sign factors depending on J, K, L and M. Starting in a local frame this becomes:

$$(-1)^{X_{JKLM}}R(flat)_{JKLM} + (-1)^{Y_{JMKL}}R(flat)_{JMKL} + (-1)^{Z_{JLMK}}R(flat)_{JLMK} = 0. \tag{3.78}$$

Expanding this out and solving for X, Y and Z we get the following:

$$(-1)^{KM}R(flat)_{JKLM} + (-1)^{ML}R(flat)_{JMKL} + (-1)^{LK}R(flat)_{JLMK} = 0. \tag{3.79}$$

Now we wish to check if this in a general frame. Again we need some identies involving the C_{JKLM}:

$$C_{LKJM} =(-1)^{K(L+J)+LJ}C_{JKLM}, \tag{3.80}$$

$$C_{JMLK} =(-1)^{L(M+K)+KM}C_{JKLM}. \tag{3.81}$$

Together these make up the previously found symmetry for $(J+K) \leftrightarrow (L+M)$. Note that this $(J \leftrightarrow L)$ and $(K \leftrightarrow M)$ symmetry is not found in $R(flat)_{JKLM}$, only the combined $(J+K) \leftrightarrow (L+M)$ symmetry.

$$\begin{aligned} C_{LMJK} =&(-1)^{M(L+J)+JL} C_{JMLK} \\ =&(-1)^{M(L+J)+JL+L(M+K)+KM} C_{JKLM} \\ =&(-1)^{(J+K)(L+M)} C_{JKLM} \end{aligned} \tag{3.82}$$

We can now check the first Bianchi identity in a general frame:

$$(-1)^{KM} R_{JKLM} + (-1)^{ML} R_{JMKL} + (-1)^{LK} R_{JLMK}, \tag{3.83}$$

$$\begin{aligned} =&(-1)^{KM}\Big[R(flat)_{JKLM} + C_{JKLM} - (-1)^{LM} C_{JKML}\Big] \\ &+ (-1)^{ML}\Big[R(flat)_{JMKL} + C_{JMKL} - (-1)^{KL} C_{JMLK}\Big] \\ &+ (-1)^{LK}\Big[R(flat)_{JLMK} + C_{JLMK} - (-1)^{MK} C_{JLKM}\Big]. \end{aligned} \tag{3.84}$$

Making use of Equations 3.80, 3.81 and 3.79 we find the following:

$$(-1)^{KM} R_{JKLM} + (-1)^{ML} R_{JMKL} + (-1)^{LK} R_{JLMK} = 0. \tag{3.85}$$

Hence we have the form of the first Bianchi identity in a general frame.

3.15 Second Bianchi identity in a local frame

We will now look at the second Bianchi identity. It will take the form:

$$(-1)^{X_{JKLMN}} R_{JKLM;N} + (-1)^{Y_{JKNLM}} R_{JKNL;M} + (-1)^{Z_{JKMNL}} R_{JKMN;L} = 0, \tag{3.86}$$

where X, Y and Z are unknowns to be determined like in previous sections. We can only treat this in a local frame, as otherwise we would need a formalism for taking covariant derivatives of non tensors like $G_{JK,LM}$. So we let $G_{MM,L} \to 0$ and $\Gamma_{MN}{}^{L} \to 0$. Second and third derivatives of G_{MN} remain non-zero. The covariant derivative of the Riemann tensor is then $R_{JKLM;N} = (-1)^{N(J+K+L+M)} R_{JKLM,N}$ and the Bianchi identity becomes:

$$\begin{aligned} &(-1)^{N(J+K+L+M)+X_{JKLMN}} R(flat)_{JKLM,N}+ \\ &(-1)^{M(J+K+N+L)+Y_{JKNLM}} R(flat)_{JKNL,M}+ \\ &(-1)^{L(J+K+M+N)+Z_{JKMNL}} R(flat)_{JKMN,L} = 0. \end{aligned} \tag{3.87}$$

Expanding this out and solving for X, Y and Z results in the second Bianchi identity taking the form:

$$(-1)^{LN}R_{JKLM;N} + (-1)^{NM}R_{JKNL;M} + (-1)^{ML}R_{JKMN;L} = 0, \tag{3.88}$$

or

$$(-1)^{N(J+K+M)}\nabla_N R_{JKLM} + (-1)^{M(J+K+L)}\nabla_M R_{JKNL} + (-1)^{L(J+K+N)}\nabla_L R_{JKMN} = 0. \tag{3.89}$$

3.16 Ricci Tensor

The Ricci tensor is the contraction of the Riemann curvature tensor, we define it to be:

$$\begin{aligned} R_{KM} =& (-1)^{KL}G^{LJ}R_{JKLM} = (-1)^{KL}R^L{}_{KLM} \\ =& (-1)^L\left[(-1)^{L(K+M)}\Gamma_{KM}{}^L{}_{,L} - (-1)^{KM}\Gamma_{KL}{}^L{}_{,M} + \Gamma_{KM}{}^R\Gamma_{RL}{}^L - (-1)^{LM}\Gamma_{KL}{}^R\Gamma_{RM}{}^L\right]. \end{aligned} \tag{3.90}$$

We can get the symmetry of this by looking using the first Bianchi identity:

$$\begin{aligned} 0 =& (-1)^{KM}R_{JKLM} + (-1)^{ML}R_{JMKL} + (-1)^{LK}R_{JLMK} \\ \therefore 0 =& (-1)^{KL}G^{LJ}\left[(-1)^{KM}R_{JKLM} + (-1)^{ML}R_{JMKL} + (-1)^{LK}R_{JLMK}\right] = 0 \\ \therefore 0 =& (-1)^{KL}G^{LJ}R_{JKLM} + (-1)^{ML+KL+KM}G^{LJ}R_{JMKL} + (-1)^{KM}G^{LJ}R_{JLMK} = 0. \end{aligned} \tag{3.91}$$

The last term is zero as the curvature tensor is graded antisymmetric on the first two indices, while the metric is graded symmetric.

$$\begin{aligned} \therefore 0 =& (-1)^{KL}G^{LJ}R_{JKLM} - (-1)^{ML+KM}G^{LJ}R_{JMLK} \\ \therefore R_{KM} =& (-1)^{KM}(-1)^{ML}G^{LJ}R_{JMLK} \\ \therefore R_{KM} =& (-1)^{KM}R_{MK} \end{aligned} \tag{3.92}$$

Thus the Ricci tensor is graded symmetric as expected. The Ricci scalar is the contraction of the Ricci tensor and is given by:

$$\begin{aligned} R =& G^{MK}R_{KM} \\ =& (-1)^{L(K+M+L)}G^{MK}\Gamma_{KM}{}^L{}_{,L} - (-1)^{KM+L}G^{MK}\Gamma_{KL}{}^L{}_{,M} \\ &+ (-1)^L G^{MK}\Gamma_{KM}{}^R\Gamma_{RL}{}^L - (-1)^{LM+L}G^{MK}\Gamma_{KL}{}^R\Gamma_{RM}{}^L. \end{aligned} \tag{3.93}$$

3.17 Contracted second Bianchi identity

Let us now look at contracting the second Bianchi identity:

$$
\begin{aligned}
0 =& G^{LJ}\Big[(-1)^{LN}R_{JKLM;N} + (-1)^{NM}R_{JKNL;M} + (-1)^{ML}R_{JKMN;L}\Big] \\
\therefore 0 =& (-1)^{KL}G^{LJ}R_{JKLM;N} + (-1)^{NM+LN+KL}G^{LJ}R_{JKNL;M} + (-1)^{ML+LN+KL}G^{LJ}R_{JKMN;L} \\
\therefore 0 =& R_{KM;N} - (-1)^{NM}R_{KN;M} + (-1)^{ML+LN+KL}G^{LJ}R_{JKMN;L}. \qquad (3.94)
\end{aligned}
$$

The negative signs come in by making use of symmetry properties of R_{JKLM}. We can contract this again with G^{MK}.

$$
\begin{aligned}
0 =& G^{MK}R_{KM;N} - (-1)^{NM}G^{MK}R_{KN;M} + (-1)^{ML+LN+KL}G^{MK}G^{LJ}R_{JKMN;L} \\
\therefore 0 =& R_{;N} - (-1)^{NM}R^{M}{}_{N;M} - (-1)^{LN}R^{L}{}_{N;L} \\
\therefore R_{;N} =& 2(-1)^{MN}R^{M}{}_{N;M} \qquad (3.95)
\end{aligned}
$$

3.18 Palatini form of Ricci scalar

It is possible to recast the Ricci scalar in the Palatini formulation similar to that of standard GR. This allows for a simpler calculation of the Ricci scalar. The starting point is the following identity:

$$
\left(\sqrt{G_{..}}\right)_{,M} = \sqrt{G_{..}}(-1)^{N}\Gamma_{MN}{}^{N}. \qquad (3.96)
$$

Here $\sqrt{G_{..}}$ refers to the square root of the super determinant of the super metric tensor G_{MN}. Equation 3.96 was determined by trial and error, and was tested via explicit calculation in the one and two coordinate cases. By considering the fact that $G^{LK}{}_{;R} = 0$ and making use of Equation 3.96 we find:

$$
\left[(-1)^{L}G^{LK}\sqrt{G_{..}}\right]_{,L} = -\sqrt{G_{..}}G^{LM}\Gamma_{ML}{}^{K}. \qquad (3.97)
$$

Noting that this must be zero under an integration as it is a total derivative, we can show that:

$$
\begin{aligned}
&(-1)^{L(K+M+L)}\sqrt{G_{..}}G^{MK}\Gamma_{KM}{}^{L}{}_{,L} - (-1)^{KM+L}\sqrt{G_{..}}G^{MK}\Gamma_{KL}{}^{L}{}_{,M} \\
&= -2(-1)^{L}\sqrt{G_{..}}G^{MK}\Gamma_{KM}{}^{R}\Gamma_{RL}{}^{L} + 2(-1)^{LM+L}\sqrt{G_{..}}G^{MK}\Gamma_{KL}{}^{R}\Gamma_{RM}{}^{L}. \qquad (3.98)
\end{aligned}
$$

Comparing Equation 3.98 to Equation 3.93 we can see that the $\sqrt{G_{..}}R$ can be written in two simplified forms:

$$
\begin{aligned}
\sqrt{G_{..}}R =& \frac{1}{2}\left[(-1)^{L(K+M+L)}\sqrt{G_{..}}G^{MK}\Gamma_{KM}{}^{L}{}_{,L} - (-1)^{KM+L}\sqrt{G_{..}}G^{MK}\Gamma_{KL}{}^{L}{}_{,M}\right], \qquad (3.99) \\
\sqrt{G_{..}}R =& -(-1)^{L}\sqrt{G_{..}}G^{MK}\Gamma_{KM}{}^{R}\Gamma_{RL}{}^{L} + (-1)^{LM+L}\sqrt{G_{..}}G^{MK}\Gamma_{KL}{}^{R}\Gamma_{RM}{}^{L}. \qquad (3.100)
\end{aligned}
$$

We now have the Ricci scalar in Palatini form. Equation 3.99 is the more useful form as the derivative of the Christoffel symbols is far easier to calculate than the product of two. This provides a neat way to reduce the amount of calculations required to get to the Ricci scalar.

3.19 Summary

This chapter has detailed the formalism for dealing with General relativity on a graded manifold. Previous work done by other authors was not quite what we required, so it was necessary to develop the framework ourselves. A significant amount of time was spent making sure everything is self-consistent and transforms correctly, the up then down summation rule that we introduced takes care of most of this. We are now in a position to start considering various supermetrics and the resulting Einstein-Hilbert actions. This will be done in the next two chapters, first with 1 property coordinate modeling electromagnetism+gravity and then two property coordinates, similar to the weak force+gravity.

Chapter 4

General Relativity with one property coordinate

In this chapter we take the formalism from the previous one and use it to construct a model with one complex property coordinate. We introduce a $U(1)$ gauge field to ensure the resulting super metric transforms correctly under local phase transformations. From this we calculate the super Ricci tensor and super Ricci scalar and the resulting field equations. The result is a unification of electromagnetism with gravity in a manner similar to Kaluza-Klein, along with a cosmological constant. Note from now on we will refer to the super metric as simply the metric, the space time 4d metric will be specified if that is what is being referred to. The same applies for the super Ricci tensor and scalar.

4.1 Notation

As we are introducing a single property coordinate ζ and its conjugate $\bar{\zeta}$, we can write the property indices as simply ζ or $\bar{\zeta}$, so for instance in the super metric component $G_{\zeta\bar{\zeta}}$. In the next chapter we will introduce more than one property coordinate and there will be the need to have internal property indices as well.

4.2 Extended Minkowski metric

Our starting point to building the metric is the following metric distance for a flat 4+2 dimensional graded manifold:

$$ds^2 = dX^A dX^B \mathcal{I}_{BA} = dx^a dx^b \eta_{ba} + \frac{1}{2} l^2 d\zeta d\bar{\zeta} - \frac{1}{2} l^2 d\bar{\zeta} d\zeta. \tag{4.1}$$

This results in the extended Minkowski metric $\mathcal{I}_{AB}$ taking the form:

$$\mathcal{I}_{AB} = \begin{pmatrix} I_{ab} & I_{a\beta} & I_{a\bar{\beta}} \\ I_{\alpha b} & I_{\alpha\beta} & I_{\alpha\bar{\beta}} \\ I_{\bar{\alpha}b} & I_{\bar{\alpha}\beta} & I_{\bar{\alpha}\bar{\beta}} \end{pmatrix} = \begin{pmatrix} \eta_{ab} & 0 & 0 \\ 0 & 0 & \frac{1}{2}l^2 \\ 0 & -\frac{1}{2}l^2 & 0 \end{pmatrix}. \tag{4.2}$$

η_{mn} is the standard space time Minkowski metric. Also notice that we have needed to introduce a length scale l here. The property coordinates are taken to be dimensionless, so to get the correct units of length^2 for ds^2 we need to introduce a scale parameter l with units of length. We can also see that the extended Minkowski metric is graded symmetric as expected, $\mathcal{I}_{AB} = (-1)^{AB}\mathcal{I}_{BA}$. It is also invariant under Lorentz transformations and global phase transformations on the property coordinate ζ. The next step is to make it invariant under local space time dependent phase transformations, which requires the introduction of a gauge field.

4.3 Frame vectors and gauge fields

Consider a spacetime dependent $U(1)$ phase transformation to the property coordinate ζ as follows:

$$x^m \to x^m, \quad \zeta \to e^{i\theta(x)}\zeta, \quad \bar{\zeta} \to e^{-i\theta(x)}\bar{\zeta}. \tag{4.3}$$

Now if we want our metric $\mathcal{I}_{AB}$ to be a tensor it has to transform correctly. Using the rule from Equation 3.23 we get:

$$\mathcal{I}_{MN} = (-1)^{R(S+N)}\partial_M{}'^R\partial_N{}'^S\mathcal{I}'_{RS}. \tag{4.4}$$

Looking at the spacetime part of this the following arises:

$$\begin{aligned}\mathcal{I}_{mn} =& (-1)^{RS}\partial_m{}'^R\partial_n{}'^S\mathcal{I}'_{RS}, \\ =& \partial_m{}'^r\partial_n{}'^s\mathcal{I}'_{rs} - \partial_m{}'^\zeta\partial_n{}'^{\bar{\zeta}}\mathcal{I}'_{\zeta\bar{\zeta}} - \partial_m{}'^{\bar{\zeta}}\partial_n{}'^\zeta\mathcal{I}'_{\bar{\zeta}\zeta}, \\ =& \eta_{mn} - \frac{1}{2}l^2\partial_m\theta e^{i\theta(x)}\zeta\partial_n\theta e^{-i\theta(x)}\bar{\zeta} + \frac{1}{2}l^2\partial_m\theta e^{-i\theta(x)}\bar{\zeta}\partial_n\theta e^{i\theta(x)}\zeta, \\ =& \eta_{mn} + l^2\partial_m\theta\partial_n\theta\bar{\zeta}\zeta. \end{aligned} \tag{4.5}$$

We are left with an extra term, so $\mathcal{I}_{AB}$ does not transform correctly under a local phase transformation in the property coordinates. This means that scalars constructed from the metric and its derivatives, for instance the Ricci scalar R, will not be invariant under local phase transformations. To deal with this we are obliged to introduce a gauge field A_m, so phase transformations in property become gauge transformations. We will achieve this through the use of frame vectors to produce a metric that does transform correctly under gauge transformations. We adopt the following upper-triangular frame vector:

$$\mathcal{E}_M{}^A = \begin{pmatrix} e_m{}^a & -ieA_m\zeta & ie\bar{\zeta}A_m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.6}$$

and its inverse:

$$E_A{}^M = \begin{pmatrix} e_a{}^m & ieA_a\zeta & -ie\bar{\zeta}A_a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.7}$$

which satisfies

$$\mathcal{E}_M{}^A E_A{}^N = \delta_M{}^N. \tag{4.8}$$

From this we can get the metric via

$$G_{MN} = (-1)^{AN}\mathcal{E}_M{}^A\mathcal{E}_N{}^B\mathcal{I}_{BA}, \tag{4.9}$$

resulting in the following metric:

$$G_{MN} = \begin{pmatrix} g_{mn} + e^2l^2A_mA_n\bar{\zeta}\zeta & -\frac{1}{2}iel^2\bar{\zeta}A_m & -\frac{1}{2}iel^2A_m\zeta \\ -\frac{1}{2}iel^2\bar{\zeta}A_n & 0 & \frac{1}{2}l^2 \\ -\frac{1}{2}iel^2A_n\zeta & -\frac{1}{2}l^2 & 0 \end{pmatrix}. \tag{4.10}$$

Note that we have introduced a gauge coupling constant e, which is labelled this way because we will find this 1 coordinate model leads to electromagnetism. This coupling constant e is different to the space time frame vectors and vielbeins given by $e_m{}^a$. The difference between the two is fairly obvious based on context so we can contend with having both. We can also see that the metric is graded symmetric, $G_{MN} = (-1)^{MN}G_{NM}$ as desired. Using this method we can also determine the inverse metric, though first we require the inverse extended Minkowski metric,

$$\mathcal{I}^{MN} = \begin{pmatrix} \eta^{mn} & 0 & 0 \\ 0 & 0 & \frac{2}{l^2} \\ 0 & -\frac{2}{l^2} & 0 \end{pmatrix}. \tag{4.11}$$

This satisfies $\mathcal{I}^{MN}\mathcal{I}_{NL} = (-1)^{ML}\delta_L{}^M$. The inverse metric is then produced via

$$G^{MN} = (-1)^{BM}\mathcal{I}^{BA}E_A{}^ME_B{}^N, \tag{4.12}$$

resulting in the following inverse metric:

$$G^{MN} = \begin{pmatrix} g^{mn} & ieA^m\zeta & -ie\bar{\zeta}A^m \\ ieA^n\zeta & 0 & \frac{2}{l^2} - e^2A^kA_k\bar{\zeta}\zeta \\ -ie\bar{\zeta}A^n & -\frac{2}{l^2} + e^2A^kA_k\bar{\zeta}\zeta & 0 \end{pmatrix}. \tag{4.13}$$

The inverse metric is also graded symmetric, $G^{MN} = (-1)^{MN}G^{NM}$. Alternatively, the inverse metric can be found from the metric by using Equations 3.34 and 3.35.

4.4 Gauge transformations on the metric

We now wish to show that the metric we have produced does indeed transform correctly under gauge transformations. A rank 2 covariant tensor transforms as given by Equation 3.23:

$$G_{MN} = (-1)^{R(S+N)}\partial_M{}'^R\partial_N{}'^S G'_{RS}.$$

We will now make a local phase transformation in property, where $\theta(x)$ is space time dependent:

$$x'^m = x^m, \quad \zeta' = e^{i\theta(x)}\zeta, \quad \bar{\zeta}' = e^{-i\theta(x)}\bar{\zeta}. \tag{4.14}$$

The Jacobian matrix of this transformation is as follows:

$$\partial_M{}'^N = \begin{pmatrix} \partial_m{}'^n & \partial_m{}'^\zeta & \partial_m{}'^{\bar{\zeta}} \\ \partial_\zeta{}'^n & \partial_\zeta{}'^\zeta & \partial_\zeta{}'^{\bar{\zeta}} \\ \partial_{\bar{\zeta}}{}'^n & \partial_{\bar{\zeta}}{}'^\zeta & \partial_{\bar{\zeta}}{}'^{\bar{\zeta}} \end{pmatrix} = \begin{pmatrix} \delta_m{}^n & i\theta_{,m}e^{i\theta}\zeta & -i\theta_{,m}e^{-i\theta}\bar{\zeta} \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-i\theta} \end{pmatrix}. \tag{4.15}$$

We also assume that under the transformation 4.14, the gauge field undergoes the standard abelian gauge transformation:

$$A'_m = A_m + \frac{1}{e}\theta_{,m}. \tag{4.16}$$

We can now test each element of G_{MN} to see that it transforms correctly:

$$\begin{aligned} G_{mn} =& (-1)^{RS}\partial_m{}'^R\partial_n{}'^S G'_{RS}, \\ =& \frac{1}{2}\partial_m{}'^r\partial_n{}'^s G'_{rs} + \partial_m{}'^r\partial_n{}'^\zeta G'_{r\zeta} + \partial_m{}'^r\partial_n{}'^{\bar{\zeta}} G'_{r\bar{\zeta}} - \partial_m{}'^\zeta\partial_n{}'^{\bar{\zeta}} G'_{\zeta\bar{\zeta}} + (m \leftrightarrow n), \\ =& \frac{1}{2}G'_{mn} + i\theta_{,n}e^{i\theta}\zeta G'_{m\zeta} - i\theta_{,n}e^{-i\theta}\bar{\zeta}G'_{m\bar{\zeta}} - \theta_{,m}e^{i\theta}\zeta\theta_{,n}e^{-i\theta}\bar{\zeta}G'_{\zeta\bar{\zeta}} + (m \leftrightarrow n), \\ =& \frac{1}{2}g_{mn} + \frac{1}{2}l^2\bar{\zeta}\zeta(e^2A_mA_n + 2eA_m\theta_{,n} + \theta_{,m}\theta_{,n}) \\ & - \frac{1}{2}el^2\theta_{,n}(A_m + \frac{1}{e}\theta_{,m})\bar{\zeta}\zeta - \frac{1}{2}el^2\theta_{,n}(A_m + \frac{1}{e}\theta_{,m})\bar{\zeta}\zeta \\ & + \frac{1}{2}l^2\theta_{,m}\theta_{,n}\bar{\zeta}\zeta + (m \leftrightarrow n), \\ =& g_{mn} + e^2l^2A_mA_n\bar{\zeta}\zeta, \text{ as required.} \end{aligned} \tag{4.17}$$

$$
\begin{aligned}
G_{m\zeta} =&(-1)^{R(S+1)}\partial_m{}'^R\partial_\zeta{}'^S G'_{RS}, \\
=&\partial_m{}'^r\partial_\zeta{}'^\zeta G'_{r\zeta} + \partial_m{}'^{\bar\zeta}\partial_\zeta{}'^\zeta G'_{\bar\zeta\zeta}, \\
=&e^{i\theta}G'_{m\zeta} - i\theta_{,m}\bar\zeta G'_{\bar\zeta\zeta}, \\
=&-\frac{1}{2}iel^2\bar\zeta(A_m + \frac{1}{e}\theta_{,m}) + \frac{1}{2}l^2 i\theta_{,m}\bar\zeta, \\
=&-\frac{1}{2}iel^2\bar\zeta A_m, \text{ as required.}
\end{aligned} \tag{4.18}
$$

$$
\begin{aligned}
G_{m\bar\zeta} =&(-1)^{R(S+1)}\partial_m{}'^R\partial_{\bar\zeta}{}'^S G'_{RS}, \\
=&\partial_m{}'^r\partial_{\bar\zeta}{}'^{\bar\zeta} G'_{r\bar\zeta} + \partial_m{}'^{\zeta}\partial_{\bar\zeta}{}'^{\bar\zeta} G'_{\zeta\bar\zeta}, \\
=&e^{-i\theta}G'_{m\bar\zeta} + i\theta_{,m}e^{i\theta}\zeta e^{-i\theta} G'_{\zeta\bar\zeta}, \\
=&-\frac{1}{2}iel^2(A_m + \frac{1}{e}\theta_{,m})\zeta + \frac{1}{2}l^2 i\theta_{,m}\zeta, \\
=&-\frac{1}{2}iel^2 A_m\zeta, \text{ as required.}
\end{aligned} \tag{4.19}
$$

$$
\begin{aligned}
G_{\zeta\bar\zeta} =&(-1)^{R(S+N)}\partial_\zeta{}'^R\partial_{\bar\zeta}{}'^S G'_{RS}, \\
=&\partial_\zeta{}'^\zeta\partial_{\bar\zeta}{}'^{\bar\zeta} G'_{\zeta\bar\zeta}, \\
=&e^{i\theta}e^{-i\theta}\frac{1}{2}l^2, \\
=&\frac{1}{2}l^2, \text{ as required.}
\end{aligned} \tag{4.20}
$$

The other parts of the metric follow by symmetry. We also see that the metric transforming correctly ensures the inverse metric also transforms correctly:

$$
\begin{aligned}
G_{MN} =&(-1)^{R(S+N)}\partial_M{}'^R\partial_N{}'^S G'_{RS}, \\
\therefore G^{NM}G_{MN} =&(-1)^{R(S+N)}G^{NM}\partial_M{}'^R\partial_N{}'^S G'_{RS}, \\
\therefore (-1)^N\delta_N{}^N =&\left[(-1)^{R(S+N)}G^{NM}\partial_M{}'^R\partial_N{}'^S\right]G'_{RS}.
\end{aligned} \tag{4.21}
$$

However we also have that $G'^{SR}G_{\prime RS} = (-1)^S\delta_S{}^S$. Hence we must have:

$$
\begin{aligned}
G'^{SR} &= (-1)^{R(S+N)}G^{NM}\partial_M{}'^R\partial_N{}'^S, \\
\therefore G'^{RS} &= (-1)^{N(M+R)}G^{MN}\partial_M{}'^R\partial_N{}'^S.
\end{aligned} \tag{4.22}
$$

and so the inverse metric G^{MN} transforms correctly if the metric G_{MN} does.

4.5 Inclusion of scalar fields

We now look to introduce scalar fields into the metric, however only in a classical sense in the form of expectation values. For one property coordinate, the only combination that is scalar is $\bar{\zeta}\zeta$, so we look to include these wherever possible into the metric, while maintaining the correct transformation properties. In a sense the inclusion of these fields can be considered similar to including curvature in property to the metric. We could consider other types of fields or quantization but for now we just wish to see what happens in the simpler classical case. The new metric takes the same form as the old one, but with extra factors included:

$$G_{MN} = \begin{pmatrix} g_{mn}(1+2c_1\bar{\zeta}\zeta) + e^2l^2A_mA_n\bar{\zeta}\zeta & -\frac{1}{2}iel^2\bar{\zeta}A_m & -\frac{1}{2}iel^2A_m\zeta \\ -\frac{1}{2}iel^2\bar{\zeta}A_n & 0 & \frac{1}{2}l^2(1+2c_2\bar{\zeta}\zeta) \\ -\frac{1}{2}iel^2A_n\zeta & -\frac{1}{2}l^2(1+2c_2\bar{\zeta}\zeta) & 0 \end{pmatrix}. \tag{4.23}$$

The factors of c_1 and c_2 represent expectation values of some scalar field, the factors of two are included to make some of the algebra a little easier. Note that this is the most general way we could have included the factors of c_i into the metric, as the coordinates are anti-commuting the combination $\bar{\zeta}\zeta$ can only be included in the places where there are no other property indices multiplying it. The factor of c_2 had to be repeated to keep the metric graded symmetric. The inverse metric can then be found:

$$G^{MN} = \begin{pmatrix} g^{mn}(1-2c_1\bar{\zeta}\zeta) & ieA^m\zeta & -ie\bar{\zeta}A^m \\ ieA^n\zeta & 0 & \frac{2}{l^2}(1-2c_2\bar{\zeta}\zeta) - e^2A^kA_k\bar{\zeta}\zeta \\ -ie\bar{\zeta}A^n & -\frac{2}{l^2}(1-2c_2\bar{\zeta}\zeta) + e^2A^kA_k\bar{\zeta}\zeta & 0 \end{pmatrix}. \tag{4.24}$$

By inspection of Equations 4.17, 4.18, 4.19 and 4.20 it can be seen that the extra factors included in Equation 4.23 do not disrupt the transformation properties of the metric. The frame vectors to produce this metric are as follows:

$$\mathcal{E}_M{}^A = \begin{pmatrix} e_m{}^a(1+c_1\bar{\zeta}\zeta) & -ieA_m\zeta & ie\bar{\zeta}A_m \\ 0 & 1+c_2\bar{\zeta}\zeta & 0 \\ 0 & 0 & 1+c_2\bar{\zeta}\zeta \end{pmatrix}. \tag{4.25}$$

The inverse frame vectors are given by:

$$E_A{}^M = \begin{pmatrix} e_a{}^m(1-c_1\bar{\zeta}\zeta) & ieA_a\zeta & -ie\bar{\zeta}A_a \\ 0 & 1-c_2\bar{\zeta}\zeta & 0 \\ 0 & 0 & 1-c_2\bar{\zeta}\zeta \end{pmatrix}. \tag{4.26}$$

4.6 Metric super-determinant

In this chapter we will later require the super-determinant, also known as the Berezinian, of the metric. The formula for the Berezinian (Berezin 1996; Deligne 1999) for a graded super matrix M of the following form:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4.27}$$

is given by

$$\mathrm{sdet}(M) = \det(A - BD^{-1}C)\ \det(D)^{-1}, \tag{4.28}$$

$$\mathrm{sdet}(M) = \det(A)\ \det(D - CA^{-1}B)^{-1}. \tag{4.29}$$

Here A and D refer to the graded even parts of the super matrix and B and C refer to the graded odd parts. Now to get the super determinant of the metric we can use the fact that $G_{MN} = (-1)^{AN}\mathcal{E}_M{}^A\mathcal{E}_N{}^B\mathcal{I}_{BA}$ to get:

$$\mathrm{sdet}(G_{..}) = \mathrm{sdet}(\mathcal{E}_.{}^.)^2\mathrm{sdet}(\mathcal{I}_{..}). \tag{4.30}$$

Here the dots are placeholders for indices, to indicate what the super determinant is applied to. This results in:

$$\mathrm{sdet}(G_{..}) = \frac{4}{l^4}\det(g_{mn})\big[1 + (8c_1 - 4c_2)\bar{\zeta}\zeta\big], \tag{4.31}$$

and also

$$\sqrt{-G_{..}} = \frac{2}{l^2}\sqrt{-g_{..}}\big[1 + (4c_1 - 2c_2)\bar{\zeta}\zeta\big]. \tag{4.32}$$

Here we have used the fairly common shorthand of an implicit determinant (extended here to sdet) when the metric is placed under a square root.

4.7 Christoffel symbols

We can now proceed to use the formalism developed in Chapter 3 to calculate the Christoffel symbols for our metric. Using Equation 3.43,

$$\Gamma_{MN}{}^P = \frac{1}{2}\Big((-1)^{MN+LM+L}G_{LM,N} + (-1)^{LN+L}G_{LN,M} - (-1)^{LM+LN+L}G_{MN,L}\Big)G^{LP},$$

as well as our metric and its inverse as defined in Equations 4.23 and 4.24, we get the following list of Christoffel symbols:

$$
\begin{aligned}
\Gamma_{mn}{}^{l} &= \Gamma^{[g]}{}_{mn}{}^{l} + e^2 l^2 (A_n F_{mk} + A_m F_{nk}) g^{kl} \bar{\zeta}\zeta/2, \\
\Gamma_{mn}{}^{\zeta} &= \frac{\zeta}{2}\left[ie\big(2A^k \Gamma^{[g]}_{mnk} - A_{m,n} - A_{n,m}\big) - 2e^2 A_m A_n - \frac{4c_1}{l^2} g_{mn}\right], \\
\Gamma_{mn}{}^{\bar{\zeta}} &= \frac{\bar{\zeta}}{2}\left[ie\big(A_{m,n} + A_{n,m} - 2A^k \Gamma^{[g]}_{mnk}\big) - 2e^2 A_m A_n - \frac{4c_1}{l^2} g_{mn}\right], \\
\Gamma_{\zeta n}{}^{l} &= \Gamma_{n\zeta}{}^{l} = \bar{\zeta}\left[iel^2 F_{kn} g^{kl}/4 - c_1 \delta_n{}^{l}\right], \\
\Gamma_{\bar{\zeta} n}{}^{l} &= \Gamma_{n\bar{\zeta}}{}^{l} = \zeta\left[iel^2 F_{kn} g^{kl}/4 + c_1 \delta_n{}^{l}\right], \\
\Gamma_{\zeta n}{}^{\zeta} &= \Gamma_{n\zeta}{}^{\zeta} = -ieA_n - \left[\frac{e^2 l^2}{4} A^k F_{kn} + ie(c_1 - 2c_2) A_n\right] \bar{\zeta}\zeta, \\
\Gamma_{\zeta n}{}^{\bar{\zeta}} &= \Gamma_{n\zeta}{}^{\bar{\zeta}} = \Gamma_{\bar{\zeta} n}{}^{\zeta} = \Gamma_{n\bar{\zeta}}{}^{\zeta} = 0, \\
\Gamma_{\bar{\zeta} n}{}^{\bar{\zeta}} &= \Gamma_{n\bar{\zeta}}{}^{\bar{\zeta}} = ieA_n - \left[\frac{e^2 l^2}{4} A^k F_{kn} + ie(2c_2 - c_1) A_n\right] \bar{\zeta}\zeta, \\
\Gamma_{\zeta\bar{\zeta}}{}^{l} &= \Gamma_{\bar{\zeta}\zeta}{}^{l} = 0, \\
\Gamma_{\zeta\bar{\zeta}}{}^{\zeta} &= -\Gamma_{\bar{\zeta}\zeta}{}^{\zeta} = -2c_2\zeta, \\
\Gamma_{\zeta\bar{\zeta}}{}^{\bar{\zeta}} &= -\Gamma_{\bar{\zeta}\zeta}{}^{\bar{\zeta}} = -2c_2\bar{\zeta}, \\
\Gamma_{\zeta\zeta}{}^{l} &= \Gamma_{\zeta\zeta}{}^{\zeta} = \Gamma_{\zeta\zeta}{}^{\bar{\zeta}} = \Gamma_{\bar{\zeta}\bar{\zeta}}{}^{l} = \Gamma_{\bar{\zeta}\bar{\zeta}}{}^{\zeta} = \Gamma_{\bar{\zeta}\bar{\zeta}}{}^{\bar{\zeta}} = 0.
\end{aligned}
\tag{4.33}
$$

We introduce the notation $\Gamma^{[g]}{}_{mn}{}^{l}$ to refer to the standard space-time Christoffel symbols. The electromagnetic field tensor is given by the standard form $F_{mn} = A_{n,m} - A_{m,n}$.

4.8 Ricci tensor and scalar

From the Christoffel symbols the Ricci tensor components can be calculated, using Equation 3.90:

$$
R_{KM} = (-1)^L \left[(-1)^{L(K+M)} \Gamma_{KM}{}^{L}{}_{,L} - (-1)^{KM} \Gamma_{KL}{}^{L}{}_{,M} + \Gamma_{KM}{}^{R} \Gamma_{RL}{}^{L} - (-1)^{LM} \Gamma_{KL}{}^{R} \Gamma_{RM}{}^{L} \right].
$$

We will now give a list of the Ricci tensor components, however only for Minkowski space time, ignoring derivatives of the space time metric g_{mn}. Calculations of the field equations and Ricci scalar were made using the full curved space time Ricci tensor components but the extra complication due to space time curvature clutters the presentation of them here and isn't particularly enlightening. We have then:

$$\begin{aligned}
R_{km} &= 4c_1 g_{km}[1-2(c_1-c_2)\bar{\zeta}\zeta]/l^2 - 4e^2(2c_1-3c_2)A_k A_m\bar{\zeta}\zeta \\
&\quad +e^2 l^2 g^{nl}\big[A_{k,n}A_{m,l} - A_{n,m}A_{l,k}\big]\bar{\zeta}\zeta/2, && (4.34)\\
R_{k\zeta} &= 2ie(2c_1-3c_2)\bar{\zeta}A_k + iel^2\bar{\zeta}F^l{}_{k,l}/4, && (4.35)\\
R_{k\bar{\zeta}} &= 2ie(2c_1-3c_2)A_k\zeta + ie\zeta l^2 F^l{}_{k,l}/4, && (4.36)\\
R_{\zeta\bar{\zeta}} &= [6c_2-4c_1+4(c_2-c_1)(3c_2-c_1)\bar{\zeta}\zeta] - l^4e^2F^{kl}F_{kl}\,\bar{\zeta}\zeta/16. && (4.37)
\end{aligned}$$

Using these we can get the raised Ricci tensor components via:

$$R^{KM} = (-1)^{L(M+N)}G^{KL}G^{MN}R_{LN}. \tag{4.38}$$

This results in raised Ricci tensor components of:

$$\begin{aligned}
R^{km} &= 4g^{mk}c_1[1+(2c_2-6c_1)\bar{\zeta}\zeta]/l^2 - e^2l^2F^k{}_lF^{ml}\bar{\zeta}\zeta/2, && (4.39)\\
R^{k\zeta} &= 4iec_1A^k\zeta/l^2 - ieF^{kl}{}_{,l}\zeta/2, && (4.40)\\
R^{k\bar{\zeta}} &= -4iec_1\bar{\zeta}A^k/l^2 + ie\bar{\zeta}F^{kl}{}_{,l}/2, && (4.41)\\
R^{\zeta\bar{\zeta}} &= 8[3c_2(1-2c_2\bar{\zeta}\zeta) - 2c_1(1-c_1\bar{\zeta}\zeta)]/l^4 \\
&\quad -e^2(4A^mA_mc_1/l^2 + F_{mn}F^{mn}/4 + A_mF^{nm}{}_{,n})\bar{\zeta}\zeta. && (4.42)
\end{aligned}$$

We can now calculate the Ricci scalar from these, using Equation 3.93:

$$R = G^{MK}R_{KM}.$$

We do this in a general frame, including space time curvature, and get the following result:

$$R = R^{[g]}(1-2c_1\bar{\zeta}\zeta) + [32c_1 - 24c_2 + (64c_1c_2 - 80{c_1}^2)\bar{\zeta}\zeta]/l^2 - e^2l^2F^{nl}F_{nl}\bar{\zeta}\zeta/4. \tag{4.43}$$

Note that the Ricci scalar is invariant under gauge transformations, which was ensured when we made our metric transform correctly as a rank 2 tensor under local gauge transformations. We are now in a position to look at the Einstein-Hilbert Lagrangian and field equations for our model.

4.9 Lagrangian for 1 property coordinate

We adopt the standard Einstein-Hilbert Lagrangian density, except with integration over property and the use of our super Ricci scalar and metric super determinant. Evaluating this results in the following Lagrangian density:

$$\mathcal{L} = \int d\zeta d\bar{\zeta}\sqrt{-G..}R = 2e^2\sqrt{-g..}\left[\frac{2(c_1-c_2)R^{[g]}}{e^2l^2} - \frac{F_{mn}F^{mn}}{4} + \frac{48(c_1-c_2)^2}{e^2l^4}\right]. \tag{4.44}$$

We can see we are left with the Einstein-Hilbert Lagrangian density for space time, the electromagnetic Lagrangian density and a cosmological term. The electromagnetic Lagrangian density has arisen purely out of the geometry introduced by including a property coordinate, effectively unifying the forces of gravity and electromagnetism. The cosmological term indicates that this model can also possibly give us an explanation of the cosmological constant as well. Comparing Equation 4.44 with the standard Einstein-Hilbert Lagrangian density for electromagnetism:

$$\mathcal{L} = \frac{1}{2\kappa}(R^{[g]} - 2\Lambda) - \frac{1}{4}F^{mn}F_{mn}, \tag{4.45}$$

where $\kappa = 8\pi G_N/c^4$ and Λ is the cosmological constant, we get the following:

$$\kappa = \frac{e^2l^2}{4(c_1-c_2)} \qquad \Lambda = \frac{12(c_2-c_1)}{l^2}. \tag{4.46}$$

Now since $\kappa > 0$, this implies $c_1 - c_2 > 0$, which implies Λ is negative. Since observations indicate that the cosmological constant is positive this is in direct conflict with observation. This one property coordinate model however was not designed to be complete, rather it demonstrates that gravity and electromagnetism can be successfully unified via the introduction of property coordinates, with the potential for an explanation of the cosmological constant. It is hoped that this problem will be alleviated by adding extra property coordinates, which will be discussed next chapter.

4.10 Field equations

To further explore our model we look at constructing field equations by considering variations of the Lagrangian density with respect to the space time metric and the gauge field. Before we proceed however, there is a detail that is not present in standard GR. Namely, we need to consider the variation of our super metric with respect to the space time metric or gauge field. First we consider the variation of the inverse super metric with respect to the contravariant gauge field A^p:

$$\delta G^{MK} = \delta A^p \begin{pmatrix} 0 & ie\zeta\delta_p{}^m & -ie\bar{\zeta}\delta_p{}^m \\ ie\zeta\delta_p{}^k & 0 & -2e^2A_p\bar{\zeta}\zeta \\ -ie\bar{\zeta}\delta_p{}^k & 2e^2A_p\bar{\zeta}\zeta & 0 \end{pmatrix}. \tag{4.47}$$

We can now look at the variation of the Lagrangian with respect to the gauge field, by using the standard field equations; these examine the variation of the Lagrangian density with respect to the super metric, and multiplying by the variation of the super metric with respect to the gauge field.

$$\int d\zeta d\bar{\zeta}\sqrt{G..}(-1)^{K+M}\big(R_{KM}-\frac{1}{2}G_{KM}R\big)\delta G^{MK}/\delta A^p \tag{4.48}$$

Evaluating this and equating to zero gives us the following result:

$$F^m{}_{p;m}=0, \tag{4.49}$$

namely the Maxwell equations in a vacuum (with no current sources). We can repeat this process with the covariant gauge field A_p, first finding the variation of the super metric with respect to the gauge field:

$$\delta G_{MK}=\delta A_p\begin{pmatrix} e^2l^2\bar{\zeta}\zeta(A_m\delta_k{}^p+A_k\delta_m{}^p) & -\frac{i}{2}e\Lambda^2\bar{\zeta}\delta_m{}^p & -\frac{i}{2}e\zeta\Lambda^2\delta_m{}^p \\ -\frac{i}{2}e\Lambda^2\bar{\zeta}\delta_k{}^p & 0 & 0 \\ -\frac{i}{2}e\zeta\Lambda^2\delta_k{}^p & 0 & 0 \end{pmatrix}. \tag{4.50}$$

We then again use the field equations to find the variation of the Lagrangian density with respect to the gauge field,

$$\int d\zeta d\bar{\zeta}\sqrt{G..}\big(R^{KM}-\frac{1}{2}G^{KM}R\big)\delta G_{MK}/\delta A_p=2F^{pm}{}_{;m}=0, \tag{4.51}$$

consistent with our first result. We can also look at variation with respect to the space time metric. The variation of the super metric is quite simple in this case. As the space time metric appears only once in the super metric we can just evaluate the following:

$$\begin{aligned}\int d\zeta d\bar{\zeta}\sqrt{G..}(R^{km}-\frac{1}{2}G^{km}R)(1+2c_1\bar{\zeta}\zeta)&\\ =&\frac{4}{l^2}(c_1-c_2)(R^{[g]km}-\frac{1}{2}g^{km}R^{[g]})-e^2(F^{kn}F^m{}_n-\frac{1}{4}F^{mn}F_{mn})\\ &-48\frac{1}{l^4}g^{km}(c_1-c_2)^2.\end{aligned} \tag{4.52}$$

Re-arranging and equating to zero gives us the following field equation:

$$R^{[g]km}-\frac{1}{2}g^{km}R^{[g]}+\frac{12(c_2-c_1)}{l^2}g^{km}=\frac{e^2l^2}{4(c_1-c_2)}T^{km}. \tag{4.53}$$

Here $T^{km}=F^{kn}F^m{}_n-\frac{1}{4}g^{km}F^{nl}F_{nl}$ is the electromagnetic stress energy tensor. We can immediately read off that:

$$\kappa=\frac{e^2l^2}{4(c_1-c_2)},\quad \Lambda=\frac{12(c_2-c_1)}{l^2}. \tag{4.54}$$

which is in agreement the result from the Lagrangian density found in Equation 4.46. The last check is to consider the variation with respect to the inverse metric, the variation of the inverse super metric is as follows:

$$\delta G^{MK} = \begin{pmatrix} 1 - 2c_1\bar{\zeta}\zeta & 0 & 0 \\ 0 & 0 & A_m A_k \bar{\zeta}\zeta \\ 0 & -A_m A_k \bar{\zeta}\zeta & 0 \end{pmatrix} \delta g^{mk}. \tag{4.55}$$

Evaluating the variation of the Lagrangian density with respect to the inverse metric results in the following:

$$\begin{aligned} \int d\zeta d\bar{\zeta}\sqrt{G..}(R_{KM} - \frac{1}{2}G_{KM}R)\delta G^{MK}/\delta g^{mk} \\ = \frac{4}{l^2}(c_1 - c_2)(R^{[g]}_{km} - \frac{1}{2}g_{km}R^{[g]}) + A_k F_m{}^n{}_{;n} + A_m F_k{}^n{}_{;n} \\ - (F_{kn}F_m{}^n - \frac{1}{4}F^{nl}F_{nl}g_{km}) - 48\frac{1}{\Lambda^4}g_{km}(c_1 - c_2)^2. \end{aligned} \tag{4.56}$$

Using Equation 4.49 we know that $F_m{}^n{}_{;n} = 0$, which results in Equation 4.53 again. It all ties together.

4.11 Matter fields

One last idea to finish up the one coordinate case is to consider the inclusion of matter fields and how they interact. While obviously quite simple with just one coordinate, we can show how the scheme could be extended to higher numbers of property coordinates. The first type we will consider are scalar fields, adhering to our previous rules regarding spin-statistics the superscalar field $\Phi(X)$ is overall Bose and thus must consist of terms containing an even number of property coordinates. Thus it must take the general form:

$$\Phi(x, \zeta, \bar{\zeta}) = U(x) + V(x)\bar{\zeta}\zeta. \tag{4.57}$$

Imposing selfduality on this we find that the field can be reduced to the form:

$$\Phi(X) = \varphi(x)(1 + \bar{\zeta}\zeta)/2. \tag{4.58}$$

ϕ carries zero charge and could be thought of as an analogue to the Higgs field, though with only one property coordinate we lack the ability to properly identify it. We now couple this to our metric, though we will drop the c_i curvature to simplify the results. A mass term in the Lagrangian of $\mu^2\varphi^2/2$ will arise through the property integral:

$$(\ell^2/2)\int d\zeta d\bar{\zeta}\sqrt{-G..}\,\mu^2\Phi^2 = \int d\zeta d\bar{\zeta}\sqrt{-g..}(1 + 2\bar{\zeta}\zeta)\mu^2\varphi^2/4. \tag{4.59}$$

The kinetic term is a bit more involved as the ζ and $\bar{\zeta}$ derivatives contribute to the mass as well, thus we consider:

$$(l^2/2)\int d\zeta d\bar{\zeta}\ \sqrt{-G..}\ G^{MN}\partial_N\Phi\,\partial_M\Phi. \tag{4.60}$$

Upon including the metric from 4.13 we find the contributions from the gauge field cancel out as required and we are left with:

$$\int d\zeta d\bar{\zeta}\sqrt{-g..}[(1+2\bar{\zeta}\zeta)\,g^{mn}\partial_n\varphi\,\partial_m\varphi/4+\bar{\zeta}\zeta\varphi^2/l^2]. \tag{4.61}$$

Thus the only way to obtain a massless scalar field is to match the kinetic mass term φ^2/l^2 from Equation 4.61 with the previously constructed mass term in Equation 4.59.

Moving on to spinor fields we now need to generalise the Dirac equation to our graded superspace. The natural way to do this is by taking $i\gamma^a e_a{}^m\partial_m$ to $i\Gamma^A E_A{}^M\partial_M$, the trick is now to determine the extended Dirac matrices Γ^M. If the Dirac operator acts on a spinorial superfield of the form $\Psi(X)=\theta\bar{\zeta}\psi(x)$, then the following representation for the Dirac matrices works:

$$\Gamma^a=\gamma^a,\ l\Gamma^{\zeta}=2i\partial/\partial\theta,\ l\Gamma^{\bar{\zeta}}=2i\theta, \tag{4.62}$$

where θ is another complex anti-commuting scalar that we eventually need to integrate over θ and $\bar{\theta}$. The action of the extended Dirac operator then yields:

$$\begin{aligned} i\Gamma^A E_A{}^M\partial_M\Psi &= \left[i\gamma^a e_a{}^m\partial_m+e\gamma^a A_a\bar{\zeta}\frac{\partial}{\partial\bar{\zeta}}+\ \frac{2}{l}(1-f\bar{\zeta}\zeta)\frac{\partial^2}{d\theta d\bar{\zeta}}\right]\theta\bar{\zeta}\psi \\ &= \theta\bar{\zeta}\gamma^a e_a{}^m(i\partial_m+eA_m)\psi-\frac{2}{l}(1-f\bar{\zeta}\zeta)\psi. \end{aligned} \tag{4.63}$$

When we include the adjoint $\bar{\Psi}\equiv-\bar{\psi}\zeta\bar{\theta}$ and integrate over $\zeta,\bar{\zeta},\theta$ and $\bar{\theta}$ we end up with the normal gauge invariant spinorial Lagrangian density:

$$\begin{aligned} \mathcal{L} &= \int(d\zeta d\bar{\zeta})(d\theta d\bar{\theta})\bar{\Psi}(X)\left[i\Gamma^A E_A{}^M\partial_M-\mathcal{M}\right]\Psi(X) \\ &= \bar{\psi}(x)\left[\gamma^a e_a{}^m(i\partial_m+eA_m)-\mathcal{M}\right]\psi(x). \end{aligned} \tag{4.64}$$

While everything works out nicely, this is obviously a very simple system. The representation of the extended Dirac matrices Γ^M will need to be revisited in order to include chirality, if we are to encompass all spin states.

4.12 Summary

In this chapter we have produced a super metric with one complex property coordinate. To enforce that it transforms correctly as a rank 2 covariant tensor under locally varying space time transformations we introduce a gauge field; these transformations are then none other than gauge transformations. We generalise the metric by including factors that act like curvature in property space and then calculate Christoffel symbols, Ricci tensor components and the Ricci super scalar. The result of this is a Einstein-Hilbert Lagrangian density that unifies gravity with electromagnetism, as well as producing a cosmological constant. This cosmological constant has the wrong sign, but this may well be remedied by additional property coordinates. We also consider variation of the Lagrangian density with respect to the metric and gauge field, to get Maxwell equations and field equations consistent with the Einstein-Hilbert Lagrangian. Finally we briefly consider matter fields, both scalar and spinor and find they work out satisfactorily. The next step is to consider a more involved model, with two property coordinates.

The work done in this chapter appears in Delbourgo and Stack (2014), though in less detail than this chapter. The Mathematica code used to generate the results is available from the UTAS digital repository, and is documented in Appendix B.

Chapter 5

General Relativity with two property coordinates

In this chapter we now consider a model with two complex coordinates, following the process from Chapter 4. We introduce an SU(2) gauge field to our 4+4 dimensional metric and calculate the resulting Ricci tensor, Ricci scalar and field equations. The result is gravity unified with SU(2) Yang-Mills, the non-abelian version of the result of the last chapter. Without including chirality however we cannot model the weak force, so rather this serves to demonstrate that our model can be extended to multiple property coordinates.

5.1 Notation

Introducing multiple property coordinates results in having to deal with property indices, we adopt the labelling from Chapter 3. Upper case Roman letters like M, N, L represent indices that run across both graded even and graded odd elements, lower case Roman letters like m, n, l represent space-time indices and Greek letters μ, ν, λ represent odd graded indices. It is also useful to note that contravariance and covariance do not apply to the property indices in the same way they do to space-time indices, we will elaborate on this shortly.

5.2 Extended Minkowski metric

Our starting point to building our metric is the following metric distance for a flat 4+4 dimensional graded manifold:

$$ds^2 = dX^M dX^N \mathcal{I}_{NM} = dx^m dx^n \eta_{nm} + \frac{1}{2} l^2 d\zeta^\mu d\zeta^{\bar{\nu}} \eta_{\bar{\nu}\mu} + \frac{1}{2} l^2 d\zeta^{\bar{\mu}} d\zeta^\nu \eta_{\nu\bar{\mu}}. \tag{5.1}$$

This results in the extended Minkowski metric taking the following form:

$$\mathcal{I}_{NM} = \begin{pmatrix} \eta_{nm} & 0 & 0 \\ 0 & 0 & \frac{1}{2}l^2\eta_{\nu\bar{\mu}} \\ 0 & \frac{1}{2}l^2\eta_{\bar{\nu}\mu} & 0 \end{pmatrix}, \tag{5.2}$$

where $\eta_{\mu\bar{\nu}} = \delta_{\mu\bar{\nu}}$, $\eta_{\bar{\nu}\mu} = \delta_{\bar{\nu}\mu}$ and $\eta_{\mu\bar{\nu}} = -\eta_{\bar{\nu}\mu}$. Note that $\mathcal{I}$ is graded symmetric,

$$\mathcal{I}_{MN} = (-1)^{MN}\mathcal{I}_{NM}. \tag{5.3}$$

The space-time piece, η_{nm} is simply the Minkowski metric, and is used to swap between contravariant and covariant indices in flat space-time. The property sector piece, $\eta_{\nu\bar{\mu}}$ is used to swap between raised and lowered property indices. As the coordinates themselves are scalar, there is no dependence on the curvature of space-time and hence the raising and lowering of property indices using $\eta_{\mu\bar{\nu}}$ and its inverse $\eta^{\mu\bar{\nu}} = \delta^{\mu\bar{\nu}}$ can be performed even in curved space-time.

Like in the one coordinate case, this flat-space metric is not invariant under space-time dependent phase transformations in property. Consider a transformation of the form:

$$x^m \to x^m, \quad \zeta^\mu \to \left(e^{i\Theta(x)}\right)^{\mu\bar{\nu}}\zeta^\nu, \quad \zeta^{\bar{\mu}} \to \zeta^{\bar{\nu}}\left(e^{-i\Theta(x)}\right)^{\nu\bar{\mu}}. \tag{5.4}$$

We again require our metric to transform as a rank 2 covariant tensor, but if we consider the following:

$$\mathcal{I}_{MN} = (-1)^{R(S+N)}\partial_M{}'^R\partial_N{}'^S\mathcal{I}'_{RS}, \tag{5.5}$$

and look at the space-time component under the property phase transformation we find:

$$\begin{aligned} \mathcal{I}_{mn} =&(-1)^{RS}\partial_m{}'^R\partial_n{}'^S\mathcal{I}'_{RS}, \\ =&\partial_m{}'^r\partial_n{}'^s\mathcal{I}'_{rs} - \frac{1}{2}\partial_m{}'^\rho\partial_n{}'^{\bar{\sigma}}\mathcal{I}'_{\rho\bar{\sigma}} - \frac{1}{2}\partial_m{}'^{\bar{\rho}}\partial_n{}'^\sigma\mathcal{I}'_{\bar{\rho}\sigma}, \\ =&\eta_{mn} - \frac{1}{2}l^2\zeta^{\bar{\mu}}\left(\partial_n e^{-i\Theta(x)}\right)^{\mu\bar{\sigma}}\delta_{\bar{\sigma}\rho}\left(\partial_m e^{i\Theta(x)}\right)^{\rho\bar{\nu}}\zeta^\nu \\ &- \frac{1}{2}l^2\zeta^{\bar{\nu}}\left(\partial_m e^{-i\Theta(x)}\right)^{\nu\bar{\rho}}\delta_{\bar{\rho}\sigma}\left(\partial_n e^{i\Theta(x)}\right)^{\sigma\bar{\mu}}\zeta^\mu. \end{aligned} \tag{5.6}$$

The last two terms do not cancel, similar to the abelian case, and so $\mathcal{I}_{MN}$ does not transform correctly as a rank 2 covariant tensor. Thus to produce our true metric G_{MN} we need to introduce a non-abelian gauge field $W_m{}^{\mu\bar{\nu}}$. Before we do this however we need to discuss the notation for matrices in the property sector.

5.3 Property indices and matrices

When considering particle physics, the adjoint is the hermitian conjugate. Casting this in terms of linear algebra, for a column vector field the adjoint is then a row vector. We adopt a similar convention here, where ζ^{μ} acts like a column vector and $\zeta^{\bar{\mu}}$ acts like a row vector. A matrix in property space then has two indices, an unbarred one and a barred one, for example a unitary transformation matrix would be represented as $U^{\mu\bar{\nu}}$ where μ and $\bar{\nu}$ are the property indices. A matrix multiplication is then done with a barred index followed by an unbarred index, for example a unitary transformation to a property index would be $\zeta'^{\mu} = U^{\mu\bar{\nu}}\zeta^{\nu}$. Lowered indices will also appear in this work, however since property coordinates can be raised and lowered freely using $\eta_{\mu\bar{\nu}}$ and $\eta^{\mu\bar{\nu}}$ an equivalent matrix with raised indices can be found. For example, consider $\zeta^{\mu}\delta_{\mu}{}^{\nu} = \zeta^{\nu}$ and $\delta^{\nu\bar{\mu}}\zeta^{\mu} = \zeta^{\nu}$, from this we can see that $\delta_{\mu}{}^{\nu}$ acts like $\delta^{\nu\bar{\mu}}$. This can be done to convert all δ's with mixed or lowered property indices to have raised ones.

$$\zeta^{\mu}\delta_{\mu}{}^{\nu} = \zeta^{\nu},\ \text{and}\ \delta^{\nu\bar{\mu}}\zeta^{\mu} = \zeta^{\nu},\ \text{gives}\ \delta_{\mu}{}^{\nu} = \delta^{\nu\bar{\mu}}. \tag{5.7}$$

$$\zeta^{\bar{\mu}}\delta_{\bar{\mu}}{}^{\bar{\nu}} = \zeta^{\bar{\nu}},\ \text{and}\ \zeta^{\bar{\mu}}\delta^{\mu\bar{\nu}} = \zeta^{\bar{\nu}},\ \text{gives}\ \delta_{\bar{\mu}}{}^{\bar{\nu}} = \delta^{\mu\bar{\nu}}. \tag{5.8}$$

$$\delta_{\mu\bar{\nu}}\delta^{\rho\bar{\nu}} = -\delta_{\mu\bar{\nu}}\delta^{\bar{\nu}\rho} = \delta_{\mu}{}^{\rho} = \delta^{\rho\bar{\mu}},\ \text{and}\ \delta^{\rho\bar{\nu}}\delta^{\nu\bar{\mu}} = \delta^{\rho\bar{\mu}},\ \text{gives}\ \delta_{\mu\bar{\nu}} = \delta^{\nu\bar{\mu}}. \tag{5.9}$$

Following from the above conventions we then have:

$$\zeta^{\nu} = \delta^{\nu\bar{\mu}}\zeta^{\mu} = \zeta^{\mu}\delta^{\nu\bar{\mu}} = \zeta^{\mu}\delta_{\mu\bar{\nu}} = \zeta_{\bar{\nu}}, \tag{5.10}$$

$$\zeta^{\bar{\nu}} = \zeta^{\bar{\mu}}\delta^{\mu\bar{\nu}} = \delta_{\nu\bar{\mu}}\zeta^{\bar{\mu}} = -\zeta_{\nu}. \tag{5.11}$$

These conditions on the property coordinates are quite general for property indices, an unbarred index $_{\mu}$ is equivalent to $^{\bar{\mu}}$ with a minus sign, $_{\bar{\mu}}$ is equivalent to $^{\mu}$ with the same sign. We can now rewrite our totally flat metric $\mathcal{I}_{NM}$ to have all the property indices raised:

$$\mathcal{I}_{NM} = \begin{pmatrix} \eta_{nm} & 0 & 0 \\ 0 & 0 & \frac{1}{2}l^2\delta^{\mu\bar{\nu}} \\ 0 & -\frac{1}{2}l^2\delta^{\nu\bar{\mu}} & 0 \end{pmatrix}. \tag{5.12}$$

This re-writing of property indices simplifies the process of dealing with the non-abelian gauge fields, which we now look to introduce.

5.4 Frame vectors and gauge fields

To get our metric to transform correctly as a rank 2 covariant tensor under local phase transformations in property we need to introduce a non-abelian gauge field $W_m{}^{\mu\bar{\nu}}$, as well as its associated gauge coupling constant g_w. Following the procedure from Chapter 4, we do this by defining frame vectors: $\mathcal{E}_M{}^A$, which take our extended Minkowski metric $\mathcal{I}_{MN}$ and produce a metric G_{MN} that transforms correctly. Again we adopt an upper-triangular frame vector of the form:

$$\mathcal{E}_M{}^A = \begin{pmatrix} e_m{}^a & -ig_w W_m{}^{\alpha\bar{\nu}}\zeta^\nu & ig_w\zeta^{\bar{\nu}}W_m{}^{\nu\bar{\alpha}} \\ 0 & \delta_\mu{}^\alpha & 0 \\ 0 & 0 & \delta_{\bar{\mu}}{}^{\bar{\alpha}} \end{pmatrix} = \begin{pmatrix} e_m{}^a & -ig_w W_m{}^{\alpha\bar{\nu}}\zeta^\nu & ig_w\zeta^{\bar{\nu}}W_m{}^{\nu\bar{\alpha}} \\ 0 & \delta^{\alpha\bar{\mu}} & 0 \\ 0 & 0 & \delta^{\mu\bar{\alpha}} \end{pmatrix}, \tag{5.13}$$

and its inverse:

$$E_A{}^M = \begin{pmatrix} e_a{}^m & ig_w W_a{}^{\mu\bar{\nu}}\zeta^\nu & -ig_w\zeta^{\bar{\nu}}W_a{}^{\nu\bar{\mu}} \\ 0 & \delta_\alpha{}^\mu & 0 \\ 0 & 0 & \delta_{\bar{\alpha}}{}^{\bar{\mu}} \end{pmatrix} = \begin{pmatrix} e_a{}^m & ig_w W_a{}^{\mu\bar{\nu}}\zeta^\nu & -ig_w\zeta^{\bar{\nu}}W_a{}^{\nu\bar{\mu}} \\ 0 & \delta^{\mu\bar{\alpha}} & 0 \\ 0 & 0 & \delta^{\alpha\bar{\mu}} \end{pmatrix}, \tag{5.14}$$

which satisfy:

$$\mathcal{E}_M{}^B E_B{}^N = \delta_M{}^N. \tag{5.15}$$

This produces the metric via $G_{MN} = (-1)^{AN} E_M{}^A E_N{}^B \mathcal{I}_{BA}$.

$$G_{MN} = \begin{pmatrix} g_{mn} + \frac{1}{2}g_w{}^2 l^2 \bar{\zeta}(W_m W_n + W_n W_m)\zeta & -\frac{1}{2}ig_w l^2(\bar{\zeta}W_m)^{\bar{\nu}} & -\frac{1}{2}ig_w l^2(W_m\zeta)^\nu \\ -\frac{1}{2}ig_w l^2(\bar{\zeta}W_n)^{\bar{\mu}} & 0 & \frac{1}{2}l^2\delta^{\nu\bar{\mu}} \\ -\frac{1}{2}ig_w l^2(W_n\zeta)^\mu & -\frac{1}{2}l^2\delta^{\mu\bar{\nu}} & 0 \end{pmatrix} \tag{5.16}$$

Note that summed property indices have been suppressed, for instance $(\bar{\zeta}W_n)^{\bar{\mu}} = \zeta^{\bar{\rho}}W_n{}^{\rho\bar{\mu}}$. We will do this wherever possible to help with readability. Now to get the inverse metric we first need $\mathcal{I}^{NM}$.

$$\mathcal{I}^{NM} = \begin{pmatrix} \eta^{nm} & 0 & 0 \\ 0 & 0 & \frac{2}{l^2}\delta^{\nu\bar{\mu}} \\ 0 & \frac{2}{l^2}\delta^{\bar{\nu}\mu} & 0 \end{pmatrix} = \begin{pmatrix} \eta^{nm} & 0 & 0 \\ 0 & 0 & \frac{2}{l^2}\delta^{\nu\bar{\mu}} \\ 0 & -\frac{2}{l^2}\delta^{\mu\bar{\nu}} & 0 \end{pmatrix} \tag{5.17}$$

The inverse metric is produced in a similar fashion to the metric via:

$$G^{MN} = (-1)^{BM}\mathcal{I}^{BA}E_A{}^M E_B{}^N. \tag{5.18}$$

This results in the following inverse metric:

$$G^{MN} = \begin{pmatrix} g^{mn} & ig_w(W^m\zeta)^\nu & -ig_w(\bar{\zeta}W^m)^{\bar{\nu}} \\ ig_w(W^n\zeta)^\mu & -g_w{}^2(W^k\zeta)^\mu(W_k\zeta)^\nu & \frac{2}{l^2}\delta^{\mu\bar{\nu}} - g_w{}^2(\bar{\zeta}W^k)^{\bar{\nu}}(W_k\zeta)^\mu \\ -ig_w(\bar{\zeta}W^n)^{\bar{\mu}} & -\frac{2}{l^2}\delta^{\nu\bar{\mu}} + g_w{}^2(\bar{\zeta}W^k)^{\bar{\mu}}(W_k\zeta)^\nu & -g_w{}^2(\bar{\zeta}W^k)^{\bar{\mu}}(\bar{\zeta}W_k)^{\bar{\nu}} \end{pmatrix}. \tag{5.19}$$

Note that the metric and its inverse are still graded symmetric, $G_{MN} = (-1)^{MN}G_{NM}$ and $G^{MN} = (-1)^{MN}G^{NM}$. The $G^{\mu\nu}$ and $G^{\bar{\mu}\bar{\nu}}$ sectors of the inverse metric are no longer 1 by 1 like they were in the abelian case, so they can be non zero without breaking the symmetry.

5.5 Gauge transformations on the metric

We now want to show that this metric transforms correctly under local phase transformations of property coordinates, which act as non-abelian gauge transformations on the gauge field. We will suppress property indices where possible, as this makes it far clearer what is going on. A rank 2 covariant tensor like the metric transforms as follows (Equation 3.23):

$$G_{MN} = (-1)^{R(S+N)} \partial_M{}'^R \partial_N{}'^S G'_{RS}. \tag{5.20}$$

We will make a local phase transformation in property, which we represent by a unitary matrix $U^{\mu\bar{\nu}}(x)$:

$$x'^m = x^m, \quad \zeta'^\mu = (U\zeta)^\mu, \quad \zeta'^{\bar{\mu}} = (\bar{\zeta}U^\dagger)^{\bar{\mu}}. \tag{5.21}$$

The Jacobian matrix of this transformation is as follows:

$$\partial_M{}'^N = \begin{pmatrix} \partial_m{}'^n & \partial_m{}'^\nu & \partial_m{}'^{\bar{\nu}} \\ \partial_\mu{}'^n & \partial_\mu{}'^\nu & \partial_\mu{}'^{\bar{\nu}} \\ \partial_{\bar{\mu}}{}'^n & \partial_{'\bar{\mu}}{}^\nu & \partial_{\bar{\mu}}{}'^{\bar{\nu}} \end{pmatrix} = \begin{pmatrix} \delta_m{}^n & (U_{,m}\zeta)^\nu & (\bar{\zeta}U^\dagger{}_{,m})^{\bar{\nu}} \\ 0 & U^{\nu\bar{\mu}} & 0 \\ 0 & 0 & U^{\dagger\mu\bar{\nu}} \end{pmatrix}. \tag{5.22}$$

Note in the above we used the fact that $\frac{\partial}{\partial\zeta^\nu}\zeta^\mu = \delta_\nu{}^\mu = \delta^{\mu\bar{\nu}}$ and $\frac{\partial}{\partial\zeta^{\bar{\nu}}}\zeta^{\bar{\mu}} = \delta_{\bar{\nu}}{}^{\bar{\mu}} = \delta^{\nu\bar{\mu}}$. We can now check the transformation properties of the metric. We will assume the standard non-abelian gauge field transformation:

$$W'{}_m = UW_mU^\dagger + \frac{i}{g_w}UU^\dagger{}_{,m} = UW_mU^\dagger - \frac{i}{g_w}U_{,m}U^\dagger. \tag{5.23}$$

Starting with G_{mn}:

$$\begin{aligned} G_{mn} =& (-1)^{RS}\partial_m{}'^R\partial_n{}'^S G'_{RS}, \\ =& \frac{1}{2}\partial_m{}'^r\partial_n{}'^s G'_{rs} + \partial_m{}'^r\partial_n{}'^\sigma G'_{r\sigma} + \partial_m{}'^r\partial_n{}'^{\bar{\sigma}} G'_{r\bar{\sigma}} - \partial_m{}'^\rho\partial_n{}'^{\bar{\sigma}} G'_{\rho\bar{\sigma}} + (m \leftrightarrow n), \\ =& \frac{1}{2}G'_{mn} - \frac{1}{2}ig_wl^2(U_{,n}\zeta)^\sigma(\bar{\zeta}W_m)'^{\bar{\sigma}} - \frac{1}{2}ig_wl^2(\bar{\zeta}U^\dagger{}_{,n})^{\bar{\sigma}}(W_m\zeta)'^\sigma \\ & - \frac{1}{2}l^2(U_{,m}\zeta)^\rho(\bar{\zeta}U^\dagger{}_{,n})^{\bar{\sigma}}\delta^{\sigma\bar{\rho}} + (m \leftrightarrow n), \\ =& \frac{1}{2}G'_{mn} + \frac{1}{2}ig_wl^2\left(\bar{\zeta}W_mU^\dagger U_{,n}\zeta\right) - \frac{1}{2}l^2\left(\bar{\zeta}U^\dagger{}_{,m}U_{,n}\zeta\right) - \frac{1}{2}ig_wl^2\left(\bar{\zeta}U^\dagger{}_{,n}UW_m\zeta\right) \\ & + (m \leftrightarrow n). \end{aligned} \tag{5.24}$$

Now consider:

$$\begin{aligned}
&\frac{1}{2}G'_{mn} + (m \leftrightarrow n) = \frac{1}{2}g_{mn} + \frac{1}{4}{g_w}^2 l^2 \left(\bar{\zeta}(W_m W_n + W_n W_m)\zeta\right)' + (m \leftrightarrow n), \\
&= \frac{1}{2}g_{mn} + \frac{1}{2}{g_w}^2 l^2 \left(\bar{\zeta} W_m W_n \zeta\right) + \frac{1}{2} i g_w l^2 \left(\bar{\zeta} W_m {U^\dagger}_{,n} U \zeta\right) + \frac{1}{2} i g_w l^2 \left(\bar{\zeta} {U^\dagger}_{,m} U W_n \zeta\right) \\
&+ \frac{1}{2} l^2 \left(\bar{\zeta} {U^\dagger}_{,m} U_{,n} \zeta\right) + (m \leftrightarrow n).
\end{aligned} \tag{5.25}$$

Putting this together we get the following:

$$\begin{aligned}
G_{mn} =& \frac{1}{2}g_{mn} + \frac{1}{2}{g_w}^2 l^2 \left(\bar{\zeta} W_m W_n \zeta\right) + (m \leftrightarrow n), \\
G_{mn} =& g_{mn} + \frac{1}{2}{g_w}^2 l^2 \bar{\zeta}(W_m W_n + W_n W_m)\zeta \text{ as required.}
\end{aligned} \tag{5.26}$$

Looking at the other elements we get:

$$\begin{aligned}
G_{m\nu} =& (-1)^{R(S+1)} {\partial_m}'^R {\partial_\nu}'^S G'_{RS}, \\
=& {\partial_m}'^r {\partial_\nu}'^\sigma G'_{r\sigma} + {\partial_m}'^{\bar{\rho}} {\partial_\nu}'^\sigma G'_{\bar{\rho}\sigma}, \\
=& -\frac{1}{2} i g_w l^2 (\bar{\zeta} W_m)'^{\bar{\sigma}} U^{\sigma\bar{\nu}} - \frac{1}{2} l^2 (\bar{\zeta} {U^\dagger}_{,m})^{\bar{\rho}} \delta^{\rho\bar{\sigma}} U^{\sigma\bar{\nu}}, \\
=& -\frac{1}{2} i g_w l^2 \left(\bar{\zeta} U^\dagger (U W_m U^\dagger + \frac{i}{e} U {U^\dagger}_{,m}) U\right)^{\bar{\nu}} - \frac{1}{2} l^2 (\bar{\zeta} {U^\dagger}_{,m} U)^{\bar{\nu}}, \\
=& -\frac{1}{2} i g_w l^2 (\bar{\zeta} U^\dagger U W_m U^\dagger U)^{\bar{\nu}} + \frac{1}{2} l^2 (\bar{\zeta} U^\dagger U {U^\dagger}_{,m} U)^{\bar{\nu}} - \frac{1}{2} l^2 (\bar{\zeta} {U^\dagger}_{,m} U)^{\bar{\nu}}, \\
=& -\frac{1}{2} i g_w l^2 (\bar{\zeta} W_m)^{\bar{\nu}} \text{ as required.}
\end{aligned} \tag{5.27}$$

$$\begin{aligned}
G_{m\bar{\nu}} =& (-1)^{R(S+1)} {\partial_m}'^R {\partial_{\bar{\nu}}}'^S G'_{RS}, \\
=& {\partial_m}'^r {\partial_{\bar{\nu}}}'^{\bar{\sigma}} G'_{r\bar{\sigma}} + {\partial_m}'^\rho {\partial_{\bar{\nu}}}'^{\bar{\sigma}} G'_{\rho\bar{\sigma}}, \\
=& {\delta_m}^r U^{\dagger\nu\bar{\sigma}} (-\frac{1}{2} i g_w l^2 (W_r \zeta)^\sigma)' + (U_{,m}\zeta)^\rho U^{\dagger\nu\bar{\sigma}} (\frac{1}{2} l^2 \delta^{\sigma\bar{\rho}}), \\
=& -\frac{1}{2} i g_w l^2 U^{\dagger\nu\bar{\sigma}} \left((U W_m U^\dagger - \frac{i}{e} U_{,m} U^\dagger) U \zeta\right)^\sigma + \frac{1}{2} l^2 U^{\dagger\nu\bar{\sigma}} \delta^{\sigma\bar{\rho}} (U_{,m}\zeta)^\rho, \\
=& -\frac{1}{2} i g_w l^2 (U^\dagger U W_m U^\dagger U \zeta)^\nu - \frac{1}{2} l^2 (U^\dagger U_{,m} U^\dagger U \zeta)^\nu + \frac{1}{2} l^2 (U^\dagger U_{,m} \zeta)^\nu, \\
=& -\frac{1}{2} i g_w l^2 (W_m \zeta)^\nu \text{ as required.}
\end{aligned} \tag{5.28}$$

$$\begin{aligned}
G_{\mu\bar{\nu}} =&(-1)^{R(S+1)}\partial_{\mu}{'^{R}}\partial_{\bar{\nu}}{'^{S}}G'_{RS},\\
=&\partial_{\mu}{'^{\rho}}\partial_{\bar{\nu}}{'^{\bar{\sigma}}}G'_{\rho\bar{\sigma}} = U^{\rho\bar{\mu}}U^{\dagger\nu\bar{\sigma}}(\frac{1}{2}l^{2}\delta^{\sigma\bar{\rho}}),\\
=&\frac{1}{2}l^{2}U^{\dagger\nu\bar{\sigma}}\delta^{\sigma\bar{\rho}}U^{\rho\bar{\mu}} = \frac{1}{2}l^{2}\delta^{\nu\bar{\mu}} \text{ as required.}
\end{aligned} \tag{5.29}$$

The other parts of the metric follow by symmetry. From Equations 4.21 and 4.22 we see that the inverse metric then also transforms correctly.

5.6 Inclusion of scalar fields

We now look to introduce scalar fields into the metric, however again we do this purely in a classical sense in the form of expectation values. For two property coordinates, the combinations of the property coordinates that produce scalar fields are $\bar{\zeta}\zeta$ and $(\bar{\zeta}\zeta)^2$, again noting that summation over property has been suppressed. We populate the metric with these fields and include corresponding expectation values ϕ_i. The fields included are not completely unrestricted though, as the metric still has to transform correctly under a gauge transformation. In a sense this process could be considered including curvature in property into the metric. The new metric takes the same form as the old one, but now with extra factors included:

$$\begin{aligned}
G_{mn} =&g_{mn}\big(1+\phi_1\bar{\zeta}\zeta+\phi_2(\bar{\zeta}\zeta)^2\big)+\frac{1}{2}e^2l^2\bar{\zeta}(W_mW_n+W_nW_m)\zeta\big(1+\phi_3\bar{\zeta}\zeta\big),\\
G_{m\nu} =&-\frac{1}{2}ig_wl^2(\bar{\zeta}W_m)^{\bar{\nu}}\big(1+\phi_4\bar{\zeta}\zeta\big),\\
G_{m\bar{\nu}} =&-\frac{1}{2}ig_wl^2(W_m\zeta)^{\nu}\big(1+\phi_5\bar{\zeta}\zeta\big),\\
G_{\mu\bar{\nu}} =&\frac{1}{2}l^2\delta^{\nu\bar{\mu}}\big(1+\phi_6\bar{\zeta}\zeta+\phi_7(\bar{\zeta}\zeta)^2\big).
\end{aligned} \tag{5.30}$$

Here we list the elements of G_{MN}, the other parts come from the graded symmetry of the metric. Note that as the property coordinates anti-commute if there is more than two ζ or $\bar{\zeta}$ in a single product they will give zero, this is why for instance $G_{m\nu}$ does not include a $\phi(\bar{\zeta}\zeta)^2$ factor. We now restrict these constants ϕ_i by considering the transformation of the metric under a gauge transformation. We make use of Equations 5.24 and 5.25 to get:

$$
\begin{aligned}
G_{mn} =& \frac{1}{2}G'_{mn} + (U_{,n}\zeta)^{\sigma}G'_{m\sigma} + (\bar{\zeta}U^{\dagger}{}_{,n})^{\bar{\sigma}}G'_{m\bar{\sigma}} - (U_{,m}\zeta)^{\rho}(\bar{\zeta}U^{\dagger}{}_{,n})^{\bar{\sigma}}G'_{\rho\bar{\sigma}} + (m \leftrightarrow n), \\
=& \frac{1}{2}g_{mn}\big(1 + \phi_1\bar{\zeta}\zeta + \phi_2(\bar{\zeta}\zeta)^2\big) + \frac{1}{2}g_w{}^2l^2\Big(\bar{\zeta}W_mW_n\zeta\Big)\big(1 + \phi_3\bar{\zeta}\zeta\big) \\
&+ \frac{1}{2}ig_wl^2\Big(\bar{\zeta}W_mU^{\dagger}{}_{,n}U\zeta\Big)\big(1 + \phi_3\bar{\zeta}\zeta\big) - \frac{1}{2}ig_wl^2\Big(\bar{\zeta}W_mU^{\dagger}{}_{,n}U\zeta\Big)\big(1 + \phi_4\bar{\zeta}\zeta\big) \\
&+ \frac{1}{2}ig_wl^2\Big(\bar{\zeta}U^{\dagger}{}_{,m}UW_n\zeta\Big)\big(1 + \phi_3\bar{\zeta}\zeta\big) - \frac{1}{2}ig_wl^2\Big(\bar{\zeta}U^{\dagger}{}_{,n}UW_m\zeta\Big)\big(1 + \phi_5\bar{\zeta}\zeta\big) \\
&+ \frac{1}{2}l^2\Big(\bar{\zeta}U^{\dagger}{}_{,m}U_{,n}\zeta\Big)\big(1 + \phi_3\bar{\zeta}\zeta\big) - \frac{1}{2}l^2\Big(\bar{\zeta}U^{\dagger}{}_{,m}U_{,n}\zeta\Big)\big(1 + \phi_4\bar{\zeta}\zeta\big) \\
&- \frac{1}{2}l^2\Big(\bar{\zeta}U^{\dagger}{}_{,n}U_{,m}\zeta\Big)\big(1 + \phi_5\bar{\zeta}\zeta\big) + \frac{1}{2}l^2(\bar{\zeta}U^{\dagger}{}_{,n}U_{,m}\zeta)\big(1 + \phi_6\bar{\zeta}\zeta\big) + (m \leftrightarrow n).
\end{aligned}
\tag{5.31}
$$

From this we can see that for G_{mn} to transform correctly we need $\phi_3 = \phi_4 = \phi_5 = \phi_6$. The other elements of G_{MN} are consistent with this, but do not provide any further restrictions. We can now relabel our constants ϕ to reflect these restrictions, the metric becomes:

$$
\begin{aligned}
G_{mn} =& g_{mn}\big(1 + c_1\bar{\zeta}\zeta + c_2(\bar{\zeta}\zeta)^2\big) + \frac{1}{2}g_w{}^2l^2\bar{\zeta}(W_mW_n + W_nW_m)\zeta\big(1 + c_3\bar{\zeta}\zeta\big), \\
G_{m\nu} =& -\frac{1}{2}ig_wl^2(\bar{\zeta}W_m)^{\bar{\nu}}\big(1 + c_3\bar{\zeta}\zeta\big), \\
G_{m\bar{\nu}} =& -\frac{1}{2}ig_wl^2(W_m\zeta)^{\nu}\big(1 + c_3\bar{\zeta}\zeta\big), \\
G_{\mu\bar{\nu}} =& \frac{1}{2}l^2\delta^{\nu\bar{\mu}}\big(1 + c_3\bar{\zeta}\zeta + c_4(\bar{\zeta}\zeta)^2\big).
\end{aligned}
\tag{5.32}
$$

5.7 Frame vectors and inverse metric with scalar fields

We now want to find the frame vectors that would produce this metric, as well as their inverses and the inverse metric. We can start by re-writing the metric as follows:

$$
\begin{aligned}
G_{mn} =& T\Big(Sg_{mn} + \frac{1}{2}g_w{}^2l^2\bar{\zeta}(W_mW_n + W_nW_m)\zeta\Big), \\
G_{m\nu} =& -T\frac{1}{2}ig_wl^2(\bar{\zeta}W_m)^{\bar{\nu}}, \\
G_{m\bar{\nu}} =& -T\frac{1}{2}ig_wl^2(W_m\zeta)^{\nu}, \\
G_{\mu\bar{\nu}} =& T\frac{1}{2}l^2\delta^{\nu\bar{\mu}},
\end{aligned}
\tag{5.33}
$$

where $T = \big(1 + c_3\bar{\zeta}\zeta + c_4(\bar{\zeta}\zeta)^2\big)$, and $S = \big(1 + [c_1 - c_3]\bar{\zeta}\zeta + [c_2 - c_4 - c_3(c_1 - c_3)](\bar{\zeta}\zeta)^2\big)$. Note that $TS = (1 + c_1\bar{\zeta}\zeta + c_2(\bar{\zeta}\zeta)^2)$ as required.

Now we need to want to find the square root and inverse of expressions like T and S, first consider:

$$(1+a_1\bar{\zeta}\zeta+a_2(\bar{\zeta}\zeta)^2)(1+b_1\bar{\zeta}\zeta+b_2(\bar{\zeta}\zeta)^2)=1+(a_1+b_1)\bar{\zeta}\zeta+(a_2+b_2+a_1b_1)(\bar{\zeta}\zeta)^2. \quad (5.34)$$

Now if we want the square root of some other factor, for instance $(1+k_1\bar{\zeta}\zeta+k_2(\bar{\zeta}\zeta)^2)$ we then let $a_i=b_i$ and require the following:

Coefficient of $\bar{\zeta}\zeta : 2a_1 = k_1 \therefore a_1 = \frac{k_1}{2}$,
Coefficient of $(\bar{\zeta}\zeta)^2 : 2a_2 + a_1^2 = k_2 \therefore a_2 = \frac{1}{2}k_2 - \frac{1}{8}{k_1}^2$.

Thus we have $\sqrt{1+k_1\bar{\zeta}\zeta+k_2(\bar{\zeta}\zeta)^2} = 1+\frac{k_1}{2}\bar{\zeta}\zeta+(\frac{1}{2}k_2-\frac{1}{8}{k_1}^2)(\bar{\zeta}\zeta)^2$.
For example $\sqrt{S} = 1+\frac{1}{2}[c_1-c_3]\bar{\zeta}\zeta+\left[\frac{1}{2}[c_2-c_4-c_3(c_1-c_3)]-\frac{1}{8}[c_1-c_3]^2\right](\bar{\zeta}\zeta)^2$.

If we wanted the inverse instead then we again consider (5.34) and require the following:

Coefficient of $\bar{\zeta}\zeta : a_1+b_1 = 0 \therefore b_1 = -a_1$,
Coefficient of $(\bar{\zeta}\zeta)^2 : a_2+b_2+a_1b_1 = 0 \therefore b_2 = {a_1}^2 - a_2$.

Thus we have $(1+a_1\bar{\zeta}\zeta+a_2(\bar{\zeta}\zeta)^2)^{-1} = (1-a_1\bar{\zeta}\zeta+({a_1}^2-a_2)(\bar{\zeta}\zeta)^2)$.

For example:

$$(\sqrt{S})^{-1} = 1-\frac{1}{2}[c_1-c_3]\bar{\zeta}\zeta+\left(\frac{1}{4}[c_1-c_3]^2-\left[\frac{1}{2}[c_2-c_4-c_3(c_1-c_3)]-\frac{1}{8}[c_1-c_3]^2\right]\right)(\bar{\zeta}\zeta)^2. \quad (5.35)$$

The frame vector that produces the metric is then given as follows:

$$\mathcal{E}_M{}^A = \sqrt{T}\begin{pmatrix} \sqrt{S}e_m{}^a & -ig_wW_m{}^{\alpha\bar{\nu}}\zeta^\nu & ig_w\zeta^{\bar{\nu}}W_m{}^{\nu\bar{\alpha}} \\ 0 & \delta^{\alpha\bar{\mu}} & 0 \\ 0 & 0 & \delta^{\mu\bar{\alpha}} \end{pmatrix}. \quad (5.36)$$

The inverse frame vector is then found to be:

$$E_A{}^M = \frac{1}{\sqrt{T}}\begin{pmatrix} (\sqrt{S})^{-1}e_a{}^m & i(\sqrt{S})^{-1}g_wW_a{}^{\mu\bar{\nu}}\zeta^\nu & -i(\sqrt{S})^{-1}g_w\zeta^{\bar{\nu}}W_a{}^{\nu\bar{\mu}} \\ 0 & \delta^{\mu\bar{\alpha}} & 0 \\ 0 & 0 & \delta^{\alpha\bar{\mu}} \end{pmatrix}. \quad (5.37)$$

The resulting inverse metric is then:

$$G^{MN} = \frac{1}{ST}\begin{pmatrix} g^{mn} & ig_w(A^m\zeta)^\nu & -ig_w(\bar\zeta W^m)^{\bar\nu} \\ ig_w(W^n\zeta)^\mu & -g_w{}^2(W^k\zeta)^\mu(W_k\zeta)^\nu & \frac{2}{l^2}S\delta^{\mu\bar\nu} - g_w{}^2(\bar\zeta W^k)^{\bar\nu}(W_k\zeta)^\mu \\ -ig_w(\bar\zeta W^n)^{\bar\mu} & -\frac{2}{l^2}S\delta^{\nu\bar\mu} + g_w{}^2(\bar\zeta W^k)^{\bar\mu}(W_k\zeta)^\nu & -g_w{}^2(\bar\zeta W^k)^{\bar\mu}(\bar\zeta W_k)^{\bar\nu} \end{pmatrix}. \tag{5.38}$$

For convenience we will also restate the metric here:

$$G_{MN} = T\begin{pmatrix} Sg_{mn} + \frac{1}{2}g_w{}^2l^2\bar\zeta(W_mW_n + W_nW_m)\zeta & -\frac{1}{2}ig_wl^2(\bar\zeta W_m)^{\bar\nu} & -\frac{1}{2}ig_wl^2(W_m\zeta)^\nu \\ -\frac{1}{2}ig_wl^2(\bar\zeta W_n)^{\bar\mu} & 0 & \frac{1}{2}l^2\delta^{\nu\bar\mu} \\ -\frac{1}{2}ig_wl^2(W_n\zeta)^\mu & -\frac{1}{2}l^2\delta^{\mu\bar\nu} & 0 \end{pmatrix}. \tag{5.39}$$

While neat in this form, the fact that S and T depend on the property coordinates means these forms are not easy to actually work with. We need to give the inverse metric explicitly expanding out the factors of S and T. Since we know that $ST = 1 + c_1\bar\zeta\zeta + c_2(\bar\zeta\zeta)^2$, then $1/(ST) = 1 - c_1\bar\zeta\zeta + (c_1^2 - c_2)(\bar\zeta\zeta)^2$. This results in the inverse metric taking the following form:

$$\begin{aligned} G^{mn} =& g^{mn}\big[1 - c_1\bar\zeta\zeta + (c_1^2 - c_2)(\bar\zeta\zeta)^2\big], \\ G^{m\nu} =& ig_w(W^m\zeta)^\nu\big[1 - c_1\bar\zeta\zeta\big], \\ G^{m\bar\nu} =& -ig_w(\bar\zeta W^m)^{\bar\nu}\big[1 - c_1\bar\zeta\zeta\big], \\ G^{\mu\nu} =& -g_w{}^2(W^k\zeta)^\mu(W_k\zeta)^\nu\big[1 - c_1\bar\zeta\zeta\big], \\ G^{\bar\mu\bar\nu} =& -g_w{}^2(\bar\zeta W^k)^{\bar\mu}(\bar\zeta W_k)^{\bar\nu}\big[1 - c_1\bar\zeta\zeta\big], \\ G^{\mu\bar\nu} =& \frac{2}{l^2}\delta^{\mu\bar\nu}\big[1 - c_3\bar\zeta\zeta + (c_3^2 - c_4)(\bar\zeta\zeta)^2\big] - g_w{}^2(\bar\zeta W^k)^{\bar\nu}(W_k\zeta)^\mu\big[1 - c_1\bar\zeta\zeta\big]. \end{aligned} \tag{5.40}$$

5.8 Metric super-determinant

We can get $\mathrm{sdet}(G_{..})$ from the frame vectors by using Equations 4.28, 4.29 and 4.30. First though we need $\mathrm{sdet}(\mathcal{I}_{..}) = \det(\eta_{mn})(\frac{1}{2}l^2)^{-4} = -\frac{16}{l^8}$. This results in:

$$\sqrt{-\mathrm{sdet}(\mathcal{I}_{BA})} = \frac{4}{l^4}. \tag{5.41}$$

Now we find the super-determinant of the frame vectors:

$$\begin{aligned} \mathrm{sdet}(\mathcal{E}_M{}^A) =& \sqrt{T}^{(4-4)}\det(e_m{}^a\sqrt{S}), \\ =& S^2\det(e_m{}^a), \\ =& \det(e_m{}^a)\big[1 + 2(c_1 - c_3)\bar\zeta\zeta + (c_1{}^2 + 2c_2 - 4c_1c_3 + 3c_3{}^2 - 2c_4)(\bar\zeta\zeta)^2\big]. \end{aligned} \tag{5.42}$$

Using Equations 5.41 and 5.42 with Equation 4.30 results in:

$$\sqrt{-G_{..}} = \frac{4}{l^4}\sqrt{-g}\big[1 + 2(c_1 - c_3)\bar{\zeta}\zeta + ({c_1}^2 + 2c_2 - 4c_1c_3 + 3{c_3}^2 - 2c_4)(\bar{\zeta}\zeta)^2\big]. \tag{5.43}$$

5.9 Christoffel symbols and Ricci tensor

The Christoffel symbols ${\Gamma_{MN}}^L$ and the contravariant Ricci tensor components R^{MN} are listed in Appendix A, they are significantly more complicated than in the abelian case. The Ricci tensor components are listed with several simplifications, space-time curvature is dropped and the constants c_i are set to $c_1 = c_2 = c_4 = 0$ and $c_3 = c$. The justification for this particular choice comes from the Lagrangian, c_3 is the only parameter that cannot be zero otherwise the Yang-Mills term disappears. Even with this simplification the Ricci tensor components are still quite long, but they are given for completeness and as a check that everything is working correctly. Once we have the Christoffel symbols and Ricci tensor components we can then move on to calculating the Lagrangian density and the field equations.

5.10 Lagrangian for 2 property coordinates

There are two ways of calculating the Lagrangian density: the first is via the Palatini form in Equation 3.99, the other way is via contraction of the Ricci tensor with the metric. Since the first of these is simpler we did that first, though when the second was done as well it was found to be consistent with the results of the Palatini form. The Lagrangian comes out to be:

$$\begin{aligned}\mathcal{L} =& \int d\zeta^2 d\bar{\zeta}^2 \sqrt{-G_{..}}R \\ =& \frac{4c_3g_w^2}{l^2}\sqrt{-g_{..}}\left[\frac{2}{c_3g_w^2l^2}\left(2c_4 - 3{c_3}^2 + 2c_1c_3 - c_2\right)R^{[g]} - \frac{1}{4}Tr\Big(\mathcal{F}^{mn}\mathcal{F}_{mn}\Big)\right. \\ &\left. + \frac{4}{c_3g_w^2l^4}\Big(- 24c_1c_2 + 38{c_1}^2c_3 + 40c_2c_3 - 110c_1{c_3}^2 + 75{c_3}^3 + 40c_1c_4 - 60c_3c_4\Big)\right].\end{aligned} \tag{5.44}$$

Comparing this with the Lagrangian density for gravity with a non-abelian gauge field,

$$\mathcal{L} = \frac{1}{2\kappa}(R^{[g]} - 2\Lambda) - \frac{1}{4}Tr\Big(\mathcal{F}^{mn}\mathcal{F}_{mn}\Big), \tag{5.45}$$

we can identify the following:

$$\kappa = 8\pi G_N/c^4 = c_3g_w^2l^2/4\left(2c_4 - 3{c_3}^2 + 2c_1c_3 - c_2\right), \tag{5.46}$$

and

$$\Lambda = \frac{24c_1c_2 - 38{c_1}^2c_3 - 40c_2c_3 + 110c_1{c_3}^2 - 75{c_3}^3 - 40c_1c_4 + 60c_3c_4}{l^2\left(2c_4 - 3{c_3}^2 + 2c_1c_3 - c_2\right)}. \tag{5.47}$$

Like in the abelian case from the previous chapter, the Yang-Mills Lagrangian density arises naturally from the geometry, as well as a cosmological constant Λ. The abelian case however only had 2 free parameters, which forced the cosmological constant to be negative. We have solved that problem, but with 4 free parameters the cosmological constant is essentially unrestricted. We are envisaging there will be additional symmetry restraints in future work, perhaps from quantisation or other restrictions that will reduce the number of free parameters. These considerations will be especially important for models involving higher numbers of property coordinates with even more possible free parameters.

5.11 Field equations

To get the field equations we find the variation of the Lagrangian density with respect to either the space-time metric g_{mn} or the gauge field W_m. To do this we need to first find the variation of the metric G_{MN}. The variation of the metric with respect to the space-time metric is simply $\delta G_{mn} = (1 + c_1\bar{\zeta}\zeta + c_2(\bar{\zeta}\zeta)^2)\delta g_{mn}$. The variation of the Lagrangian density with respect to the space-time metric is then given by:

$$
\begin{aligned}
\delta\mathcal{L}/\delta g_{mn} =& \int d^2\zeta d^2\bar{\zeta}\sqrt{-G_{..}}(R^{mn} - \frac{1}{2}G^{mn}R)\delta G_{mn}/\delta g_{mn},\\
=&\frac{1}{l^4}\left(16c_1c_3 - 8c_2 - 24{c_3}^2 + 16c_4\right)\left(R^{[g]\,mn} - \frac{1}{2}g^{mn}R^{[g]}\right) - 2\frac{1}{l^2}g_w^2 c_3 Tr(\mathcal{T}^{mn})\\
&+\frac{16}{l^6}(12c_1c_2 - 19{c_1}^2c_3 - 20c_2c_3 + 55c_1{c_3}^2 - \frac{75}{2}{c_3}^3 - 20c_1c_4 + 30c_3c_4)g^{mn}.
\end{aligned}
\tag{5.48}
$$

Where $\mathcal{T}^{mn} = \mathcal{F}^{ml}\mathcal{F}^{n}{}_{l} - \frac{1}{4}g^{mn}F^{ls}F_{ls}$ is the non abelian stress energy tensor for our gauge field W. Equating this to zero gives our field equations for the space-time metric:

$$
\begin{aligned}
R^{[g]\,mn} - \frac{1}{2}g^{mn}R^{[g]} + \frac{2}{l^2}&\frac{(24c_1c_2 - 38{c_1}^2c_3 - 40c_2c_3 + 110c_1{c_3}^2 - 75{c_3}^3 - 40c_1c_4 + 60c_3c_4)}{4c_1c_3 - 2c_2 - 6{c_3}^2 + 4c_4}g^{mn},\\
&= \frac{1}{4}l^2\frac{g_w^2c_3}{2c_1c_3 - c_2 - 3{c_3}^2 + 2c_4}Tr(\mathcal{T}^{mn}).
\end{aligned}
\tag{5.49}
$$

which is consistent with the κ and Λ from the Lagrangian above.

Now to consider variation with respect to the gauge field. First we express our gauge field in terms of a basis, ${W_m}^{\bar{\mu}\nu} = {W_m}^i{\tau_i}^{\bar{\mu}\nu}$, where $\underline{\tau} = (I, \underline{\sigma})$. We then consider the variation of our metric G_{MN} with respect to ${W_p}^i$.

$$
\begin{aligned}
\delta G_{mn} =&\frac{1}{2}g_w^2 l^2\bar{\zeta}\left({\delta_m}^p\tau_iW_n + {\delta_n}^pW_m\tau_i + {\delta_n}^p\tau_iW_m + {\delta_m}^pW_n\tau_i\right)\zeta\left(1 + c_3\bar{\zeta}\zeta\right)\delta {W_p}^i,\\
\delta G_{m\nu} =& -\frac{1}{2}g_w^2(\bar{\zeta}\tau_i)^{\bar{\nu}}\left(1 + c_3\bar{\zeta}\zeta\right){\delta_m}^p\delta {W_p}^i,\\
\delta G_{m\bar{\nu}} =& -\frac{1}{2}g_w^2(\tau_i\zeta)^{\nu}\left(1 + c_3\bar{\zeta}\zeta\right){\delta_m}^p\delta {W_p}^i.
\end{aligned}
$$

The variation of the Lagrangian density with respect to ${W_p}^i$ is then:

$$\begin{aligned}\delta\mathcal{L}/\delta {W_p}^i &= \int d^2\zeta d^2\bar{\zeta}\sqrt{-G_{..}}(R^{MN} - \frac{1}{2}G^{MN}R)\delta G_{NM}/\delta {W_p}^i, \\ &= 8c_3\frac{1}{l^2}\sqrt{g_{..}}\Big[(W_{m,p} - W_{p,m})^{,m} + 2ig_w[W^m, W_{p,m}] + ig_w[W_{m,p}, W^m] \\ &+ ig_w[{W^m}_{,m}, W_p] + g_w^2(W^m W_m W_p - 2W^m W_p W_m + W_p W^m W_m)\Big]^i. \qquad (5.50)\end{aligned}$$

Equating this to zero gives $\mathcal{D}^m\mathcal{F}_{mn} = 0$, which is the non abelian version of the Maxwell equations in free space.

5.12 Matter fields

Now we check that scalar and spinor source fields interact correctly. With two coordinates we can regard ζ^1 having the property of neutrinicity while ζ^2 has the property of electricity, just to get an idea of the properties of the resulting superfield expansions. Starting with scalar superfields and ignoring gauging and curvature, the anti selfdual scalar superfield admits an expansion containing a singlet Y and a triplet $\underline{Z}$ of SU(2):

$$\begin{aligned}\sqrt{2}\Phi &= Y[1 - (\bar{\zeta}\zeta)^2/2] + Z^0(\zeta^{\bar{1}}\zeta^1 - \zeta^{\bar{2}}\zeta^2) + Z^+\zeta^{\bar{1}}\zeta^2 + Z^-\zeta^{\bar{2}}\zeta^1 \\ &= Y[1 - (\bar{\zeta}\zeta)^2/2] + \bar{\zeta}\underline{Z}.\underline{\sigma}\zeta. \qquad (5.51)\end{aligned}$$

Using the identity that $\int d^2\zeta d^2\bar{\zeta}\ (\bar{\zeta}A\zeta)(\bar{\zeta}B\zeta) = \mathrm{Tr}(A)\mathrm{Tr}(B) - \mathrm{Tr}(AB)$, we find that:

$$\int d^2\zeta d^2\bar{\zeta}\ \Phi^2(X) = -(Y^2 + \underline{Z}.\underline{Z}). \qquad (5.52)$$

Next we include curvature into our metric, but to keep it simple we will ignore the curvature constants other than $c_3 = c$, to focus on the interaction with the gauge field. In this case the Berezinian reduces to:

$$\sqrt{G_{..}} \to 4[1 - 2c\bar{\zeta}\zeta + 3(c\bar{\zeta}\zeta)^2]/l^2. \qquad (5.53)$$

The kinetic Lagrangian then reduces to the following:

$$\begin{aligned}\int d^2\zeta d^2\bar{\zeta}\sqrt{G_{..}}G^{MN}\partial_N\Phi\partial_M\Phi &= -(1 - 3c^2)\partial^m Y\partial_m Y \qquad (5.54) \\ &\quad -\mathrm{Tr}\big[(\partial^m Z - ig_w[W^m, Z]).(\partial_m Z - ig_w[W_m, Z])\big].\end{aligned}$$

Apart from a trivial renormalisation of the Y-field, this is exactly what we would expect for the interaction of the four scalar fields with the gauge field.

Before we get started on spinor fields, we first must generalise the Dirac equation by taking $i\gamma^a {e_a}^m\partial_m$ to $i\Gamma^A {E_A}^M\partial_M$, requiring a set of extended Dirac matrices Γ^M. If we introduce auxiliary anti-commuting scalars θ^α, then one possible representation of the extended Dirac matrices is:

$$\Gamma^m = \gamma^m \quad \Gamma^\mu = \theta^\mu \quad \Gamma^{\bar{\mu}} = \partial/\partial\theta^\mu,$$

if they act on singlets $\Theta = \theta^1\theta^2 \dots \theta^N$. This representation results in $[\Gamma^\alpha, \Gamma^{\bar{\beta}}] = (1+\sigma_3)\delta_\beta{}^\alpha$, when projected on to the singlet Θ. We can now construct our anti selfdual fermionic superfield and its adjoint in flat space:

$$\Psi = (\bar{\zeta}\psi + \psi^c\zeta)(1-\bar{\zeta}\zeta)\Theta/2, \quad \bar{\Psi} = (-\bar{\psi}\zeta + \overline{\zeta\psi^c})(1-\bar{\zeta}\zeta)\overline{\Theta}/2, \tag{5.55}$$

where $\bar{\zeta}\psi \equiv \zeta^{\bar{1}}\psi^1 + \zeta^{\bar{2}}\psi^2$ and ψ^c is the charge conjugate of ψ. Note these contain the singlets Θ and $\overline{\Theta}$ to ensure our representation of the extended Dirac matrices Γ^M act correctly. The kinetic and mass terms are produced by integrating over ζ and θ:

$$\int d^2\theta d^2\bar{\theta} d^2\zeta d^2\bar{\zeta}\, \bar{\Psi} i\gamma.\partial\Psi = \int d^2\zeta d^2\bar{\zeta}\, (\bar{\zeta}\zeta)(1-2\bar{\zeta}\zeta)[\bar{\psi}i\gamma.\partial\psi + \overline{\psi^c}i\gamma.\partial\psi^c]/4 = -2\bar{\psi}i\gamma.\partial\psi, \tag{5.56}$$

$$\int d^2\theta d^2\bar{\theta} d^2\zeta d^2\bar{\zeta}\, \bar{\Psi}\Psi = \int d^2\zeta d^2\bar{\zeta}\, (\bar{\zeta}\zeta)(1-2\bar{\zeta}\zeta)[\bar{\psi}\psi + \overline{\psi^c}\psi^c]/4 = -2\bar{\psi}\psi. \tag{5.57}$$

We can now include curvature in our super metric through the frame vectors, though for simplicity we will only include $c_3 = c$ and drop the other c_i to focus on the gauge fields. This produces:

$$E_A{}^M\partial_M = [1 + c\bar{\zeta}\zeta/2](e_a{}^m\partial_m + ig_w(W_a\zeta)^\mu\partial_\mu - ig_w(\bar{\zeta}W_a)^{\bar{\mu}}\partial_{\bar{\mu}}). \tag{5.58}$$

There is a derivative over property introduced by the gauge field, along with a compensating property factor. The net result of all this is:

$$-\int d^2\zeta d^2\bar{\zeta} (\det E)\bar{\Psi}i\Gamma^A E_A{}^M\partial_M\Psi = \frac{8\sqrt{g_{\cdot\cdot}}}{l^4}(1-\frac{c}{4})\left[\bar{\psi}\gamma.(i\partial - g_w W)\psi + \overline{\psi^c}\gamma.(i\partial + g_w W)\psi^c\right]. \tag{5.59}$$

This is exactly as we anticipated, and confirms the fact that the frame vector and resulting metric have the correct forms for both scalar and spinor source fields. Note that we have not mentioned chirality here, this will require a revisiting of this work in the future.

5.13 Summary

In this chapter we have repeated the work of Chapter 4, but with two property coordinates. We started with a flat space-time-property super metric and then introduced a non abelian gauge field W to ensure it transformed correctly under locally varying non abelian phase transformations in property. We then generalised the metric by including as many scalar factors of $\bar{\zeta}\zeta$ as we could without breaking the transformation properties and calculated the resulting Christoffel symbols, Ricci tensor and Ricci scalar. These were used to produce the Lagrangian density for our model, which included gravity, a cosmological constant and the

Yang-Mills Lagrangian all arising naturally from the geometry. Since the metric allowed for 4 free parameters c_i, the cosmological constant ends up being unrestricted. We envisage that further symmetry restraints introduced by quantisation or other means will reduce the degrees of freedom present. We also then checked that the variation of the Lagrangian with respect to the space-time metric correctly produced field equations consistent with our Lagrangian density, while variation with respect to the gauge field produced the non abelian version of the Maxwell equations in free space. Lastly we showed that scalar and spinor superfields interact correctly with the gauge field introduced through the metric.

The work done in this Chapter is will appear in Stack and Delbourgo (2015), which is currently under review. The Mathematica code used to generate the results is available from the UTAS digital repository, and is documented in Appendix B.

Chapter 6

Resume and Conclusions

While much of the work is so far preliminary, it does seem like the theory of everything presented in this thesis it is quite possibly viable and deserves further study. The field expansions and mass matrices in Chapter 2 had been done before by Delbourgo (2006b), but determining the conditions on the expectation values and the use of Mathematica to perform the calculations is new to this work. Delbourgo (2006a) makes steps towards what is achieved in Chapters 3 and 4, but with a different formalism and without the use of Mathematica. This means that Chapters 3, 4 and 5 represent original work.

In Chapter 2 we introduced property coordinates to keep track of the "what" that is left out of space-time's "when" and "where". We find that we need 5 of these coordinates and their conjugates to produce all the observed particles in the standard model. After performing symmetry reductions we then we constructed explicit superfield expansions in these coordinates. We obtain non zero expectation values for the 9 Higgs-like fields via spontaneous symmetry breaking, this produces an algebraic set of conditions on the expectation values. While the concept had been presented before the explicit calculation is original to this work. The Yukawa interaction can then be used to give particles masses, producing mass matrices for the standard model particles. Some promising preliminary numerical analysis was performed, indicating the model may be viable. This leaves opportunity for future study of the system of conditions and mass matrices.

Chapter 3 considers what effect adding these coordinates would have to the formalism of general relativity. Notation is carefully defined, and general relativity is built up step-by-step starting from vectors and differentiation. Various checks are performed like the symmetry of the Riemann curvature tensor and the Bianchi identities to ensure that everything is working consistently. The end result of this chapter is that we have extended GR versions of the Ricci tensor and Ricci scalar, as well as the Palatini form of the Ricci scalar for use in the following chapters.

In Chapter 4 we take the formalism from Chapter 3 and apply it to the case of 1 property coordinate. The starting point is an extended Minkowski metric that is made to transform correctly under a local U(1) phase transformation in property by including a gauge field. This metric is then generalised with scalar terms and plugged into the formalism of Chapter 3. The resulting Einstein-Hilbert Lagrangian results in a unification of gravity with electromagnetism, along with a cosmological constant of the wrong sign. The fact that the cosmological constant came out incorrectly wasn't a huge concern as this was only a toy model to test the idea. The field equations are consistent with the Lagrangian and also produce the Maxwell equations in free space, indicating everything is working correctly.

Chapter 5 repeats the work of Chapter 4, but with 2 property coordinates. The same process is followed except with the additional complication of a non abelian gauge field introduced by an SU(2) local phase transformation in property. The resulting Einstein-Hilbert Lagrangian unifies gravity with an SU(2) Yang-Mills gauge field, and again produces a cosmological constant. This time however the cosmological constant is not restricted, with four free parameters from the metric rather than two in the previous chapter. While this solves the issue of the wrong sign, it does leave open the issue of having too much freedom, especially if more property coordinates are included. It is possible that additional constraints are required, perhaps through quantisation or other symmetry restrictions. The field equations are again consistent with the Lagrangian, again also producing the non abelian version of the Maxwell equations in free space. To properly model the weak force however, chirality is required. This is a challenge for future work.

The fact that the results of Chapters 4 and 5 came out so neatly was pleasantly surprising, we had little idea what the result would be when we started. Much of the theory came together quite neatly from the original premise rather than needing to be forced into it, which is a promising indication that further work may be rewarded as well. There is still plenty of work to be done; numerical work on the Chapter 2 content, chirality, moving to GR with 3+ coordinates, quantisation of the metric, more general symmetry symmetry constraints on the metric, further development of the Mathematica code as well as I'm sure unforeseen future issues. I believe what we have achieved here is a demonstration that the theory is viable and is worthy of further consideration amongst the many other attempts to go beyond the Standard Model and unify the known forces.

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Appendix A

Two property coordinate model

Here we list the Christoffel symbols $\Gamma_{MM}{}^{L}$ and Ricci tensor components R^{MN} as calculated for the two property coordinate model in Chapter 5. Note that $F_{mn} = W_{n,m} - W_{m,n}$ and $\mathcal{F}_{mn} = F_{mn} - ig_w[W_m, W_n]$ is the non abelian field tensor.

A.1 Christoffel symbols

Using Equation 3.43 and our metric and its inverse as defined in Equations 5.32 and 5.40 we end up with the following list of Christoffel symbols:

$\Gamma_{mn}{}^{l} = \Gamma^{[g]}{}_{mn}{}^{l} - \frac{1}{4}g_w{}^2 l^2 g^{lk}\big(1 + (c_3 - c_1)\bar{\zeta}\zeta\big)\bar{\zeta}[W_m W_{n,k} - W_m W_{k,n} + W_{m,k}W_n - W_{k,m}W_n +$
$ig_w W_m W_n W_k - ig_w W_k W_m W_n + (m \leftrightarrow n)]\zeta$

$\Gamma_{mn}{}^{\lambda} = ig_w\Gamma^{[g]}{}_{mn}{}^{k}(W_k\zeta)^{\lambda} - \frac{1}{l^2}g_{mn}\zeta^{\lambda}[c_1 + (2c_2 - c_1c_3)\bar{\zeta}\zeta] - \frac{1}{4}g_w{}^2 l^2 (A^k\zeta)^{\lambda}\bar{\zeta}\big[g_w W_k W_m W_n -$
$g_w W_m W_n W_k + iW_m W_{n,k} - iW_m W_{k,n} + iW_{m,k}W_n - iW_{k,m}W_n + (m \leftrightarrow n)\big]\zeta$
$- \frac{1}{2}c_3 g_w{}^2\bar{\zeta}(W_m W_n + W_n W_m)\zeta\zeta^{\lambda} - \frac{1}{2}g_w\big[(g_w W_m W_n + iW_{m,n})\zeta + (m \leftrightarrow n)\big]^{\lambda}$

$\Gamma_{mn}{}^{\bar{\lambda}} = -ig_w\Gamma^{[g]}{}_{mn}{}^{k}(\bar{\zeta}W_k)^{\bar{\lambda}} - \frac{1}{l^2}g_{mn}\zeta^{\bar{\lambda}}[c_1 + (2c_2 - c_1c_3)\bar{\zeta}\zeta] + \frac{1}{4}g_w{}^2 l^2 (\bar{\zeta}A^k)^{\bar{\lambda}}\bar{\zeta}\big[(g_w W_k W_m W_n -$
$g_w W_m W_n W_k + iW_m W_{n,k} - iW_m W_{k,n} + iW_{m,k}W_n - iW_{k,m}W_n + (m \leftrightarrow n)\big]\zeta$
$- \frac{1}{2}c_3 g_w{}^2\bar{\zeta}(W_m W_n + W_n W_m)\zeta\zeta^{\bar{\lambda}} - \frac{1}{2}g_w\big[\bar{\zeta}(g_w W_m W_n - iW_{m,n}) + (m \leftrightarrow n)\big]^{\bar{\lambda}}$

$\Gamma_{m\nu}{}^{l} = -\frac{1}{2}\delta_m{}^{l}\zeta^{\bar{\nu}}[c_1 + (2c_2 - c_1{}^2)(\bar{\zeta}\zeta)] - \frac{1}{4}g_w l^2[\bar{\zeta}\mathcal{F}_m{}^{l}]^{\bar{\nu}}[1 + (c_3 - c_1)(\bar{\zeta}\zeta)]$

$\Gamma_{m\bar{\nu}}{}^{l} = \frac{1}{2}\delta_m{}^{l}\zeta^{\nu}[c_1 + (2c_2 - c_1^2)(\bar{\zeta}\zeta)] - \frac{1}{4}g_w l^2[\mathcal{F}_m{}^{l}\zeta]^{\nu}[1 + (c_3 - c_1)(\bar{\zeta}\zeta)]$

$\Gamma_{m\nu}{}^{\lambda} = \frac{i}{2}g_w[\bar{\zeta}W_m]^{\bar{\nu}}\zeta^{\lambda}c_3(1 - c_3(\bar{\zeta}\zeta)) + \frac{i}{2}g_w\zeta^{\bar{\nu}}[W_m\zeta]^{\lambda}\bigg(c_3 - c_1 - (2c_2 + c_3{}^2 - c_1{}^2)(\bar{\zeta}\zeta)\bigg)$
$- ig_w(W_m)^{\lambda\bar{\nu}}[1 - c_4(\bar{\zeta}\zeta)^2] - \frac{1}{4}g_w{}^2 l^2[W^k\zeta]^{\lambda}[\mathcal{F}_{mk}]^{\bar{\nu}}[1 + (c_3 - c_1)(\bar{\zeta}\zeta)]$

$$\Gamma_{m\bar{\nu}}{}^{\bar{\lambda}} = -\tfrac{i}{2}g_w\zeta^{\bar{\lambda}}[W_m\zeta]^{\nu}c_3[1-c_3(\bar{\zeta}\zeta)] - \tfrac{i}{2}g_w[\bar{\zeta}W_m]^{\bar{\lambda}}\zeta^{\nu}\Big(c_3 - c_1 - (2c_2 + c_3{}^2 - c_1{}^2)(\bar{\zeta}\zeta)\Big)$$
$$+ ig_w(W_m)^{\nu\bar{\lambda}}[1-c_4(\bar{\zeta}\zeta)^2] + \tfrac{1}{4}g_w{}^2l^2[\bar{\zeta}W^k]^{\bar{\lambda}}[\mathcal{F}_{mk}\zeta]^{\nu}[1+(c_3-c_1)(\bar{\zeta}\zeta)]$$

$$\Gamma_{m\nu}{}^{\bar{\lambda}} = -\tfrac{1}{4}g_w{}^2l^2[\bar{\zeta}W^k]^{\bar{\lambda}}[\bar{\zeta}\mathcal{F}_{km}]^{\bar{\nu}} + \tfrac{1}{2}ig_w\Big(\zeta^{\bar{\nu}}[\bar{\zeta}W_m]^{\bar{\lambda}}(c_1-c_3) - \zeta^{\bar{\lambda}}[\bar{\zeta}W_m]^{\bar{\nu}}c_3\Big)$$

$$\Gamma_{m\bar{\nu}}{}^{\lambda} = \tfrac{1}{4}g_w{}^2l^2[W^k\zeta]^{\lambda}[\mathcal{F}_{km}\zeta]^{\nu} + \tfrac{1}{2}ig_w\Big(\zeta^{\nu}[W_m\zeta]^{\lambda}(c_1-c_3) - \zeta^{\lambda}[W_m\zeta]^{\nu}c_3\Big)$$

$$\Gamma_{\mu\nu}{}^{\lambda} = -\tfrac{1}{2}\zeta^{\bar{\nu}}(c_3{}^2-2c_4)(\bar{\zeta}\zeta)\delta^{\lambda\bar{\mu}} - \tfrac{1}{2}\delta^{\lambda\bar{\nu}}\zeta^{\bar{\mu}}[c_3-(c_3{}^2-2c_4)(\bar{\zeta}\zeta)] + \tfrac{1}{2}c_3\delta^{\lambda\bar{\mu}}\zeta^{\bar{\nu}}$$

$$\Gamma_{\bar{\mu}\bar{\nu}}{}^{\bar{\lambda}} = \tfrac{1}{2}\zeta^{\nu}\delta^{\mu\bar{\lambda}}(c_3{}^2-2c_4)(\bar{\zeta}\zeta) + \tfrac{1}{2}\delta^{\nu\bar{\lambda}}\zeta^{\mu}[c_3-(c_3{}^2-2c_4)(\bar{\zeta}\zeta)] - \tfrac{1}{2}c_3\delta^{\mu\bar{\lambda}}\zeta^{\nu}$$

$$\Gamma_{\mu\bar{\nu}}{}^{l} = -\tfrac{i}{2}g_wl^2c_4\Big(\zeta^{\bar{\mu}}[W^l\zeta]^{\nu} - [\bar{\zeta}W^l]^{\bar{\mu}}\zeta^{\nu}\Big)(\bar{\zeta}\zeta)$$

$$\Gamma_{\mu\bar{\nu}}{}^{\lambda} = \tfrac{1}{2}(c_3{}^2-2c_4)(\bar{\zeta}\zeta)\zeta^{\nu}\delta^{\lambda\bar{\mu}} + \tfrac{1}{2}(c_3{}^2-2c_4)(\bar{\zeta}\zeta)\zeta^{\lambda}\delta^{\nu\bar{\mu}} - \tfrac{1}{2}c_3(\delta^{\nu\bar{\mu}}\zeta^{\lambda} + \delta^{\lambda\bar{\mu}}\zeta^{\nu})$$

$$\Gamma_{\mu\bar{\nu}}{}^{\bar{\lambda}} = \tfrac{1}{2}(c_3{}^2-2c_4)(\bar{\zeta}\zeta)\zeta^{\bar{\mu}}\delta^{\nu\bar{\lambda}} + \tfrac{1}{2}(c_3{}^2-2c_4)(\bar{\zeta}\zeta)\zeta^{\bar{\lambda}}\delta^{\nu\bar{\mu}} - \tfrac{1}{2}c_3(\delta^{\nu\bar{\mu}}\zeta^{\bar{\lambda}} + \delta^{\nu\bar{\lambda}}\zeta^{\bar{\mu}})$$

$$\Gamma_{\mu\nu}{}^{l} = \Gamma_{\mu\nu}{}^{\bar{\lambda}} = \Gamma_{\bar{\mu}\bar{\nu}}{}^{l} = \Gamma_{\bar{\mu}\bar{\nu}}{}^{\lambda} = 0$$

A.2 Ricci tensor

We can calculate the components of the Ricci tensor R_{MN} by using Equation 3.90 and the Christoffel symbols from the previous section. The components of the raised Ricci tensor R^{MN} can then be found by considering by using Equation 4.38. We will need to simplify them to include them here, as they are quite long otherwise. First we drop space-time curvature, secondly we let $c_1 = c_2 = c_4 = 0$ and $c_3 = c$. This choice of constants still retains most of the structure of the Ricci tensor components, but greatly simplifies the number of terms.

$$R^{mn} = -\tfrac{1}{4}l^2[\bar{\zeta}(\mathcal{F}^m{}_k\mathcal{F}^{nk}g_w^2)\zeta] - c^2(\bar{\zeta}\zeta)[\bar{\zeta}(W^mW^ng_w^2)\zeta]$$
$$+ \tfrac{1}{4}cl^2\Big([\bar{\zeta}W_kg_w\zeta] - (\bar{\zeta}\zeta)\mathrm{Tr}(W_kg_w)\Big)[\bar{\zeta}(W^kW^mW^ng_w^3 + W^mW^nW^kg_w^3 - 2W^mW^kW^ng_w^3 -$$
$$i(F^{mk}W^ng_w^2 - W^mF^{nk}g_w^2))\zeta] - c^2\Big([\bar{\zeta}W^mg_w\zeta] - (\bar{\zeta}\zeta)\mathrm{Tr}(W^mg_w)\Big)[\bar{\zeta}W^ng_w\zeta]$$
$$- \tfrac{i}{4}cl^2(\bar{\zeta}\zeta)[\bar{\zeta}(F^m{}_kW^kW^ng_w^3 - W^mW_kF^{nk}g_w^3 - i(F^m{}_kF^{nk}g_w^2 - W_kW^mW^nW^kg_w^4$$
$$+ W^mW_kW^kW^ng_w^4))\zeta] + (m \leftrightarrow n)$$

$$R^{\mu\nu} = \tfrac{1}{4}[W_{k,m}g_w\zeta]^{\mu}[(F^{km}g_w - 2ig_w^2[W^k,W^m])\zeta]^{\nu} + \tfrac{1}{4}[W_kW_mg_w^2\zeta]^{\mu}[[W^k,W^m]g_w^2\zeta]^{\nu}$$
$$- \tfrac{i}{2}[W_kg_w\zeta]^{\mu}[(2W^k{}_{,m}W^mg_w^2 - 2W_mW^{k,m}g_w^2 - W_l{}^{,k}W^lg_w^2 - W_l{}^{,l}W^kg_w^2 + W_lW^{l,k}g_w^2$$
$$+ W^kW_l{}^{,l}g_w^2 - 2iW_mW^kW^mg_w^3 + iW_{l,m}{}^{,l}g_w - iW^k{}_{,l}{}^{,l}g_w + iW_lW^lW^kg_w^3 + iW^kW_lW^lg_w^3))\zeta]^{\nu}$$
$$- 9c^2\tfrac{1}{l^4}\zeta^{\mu}\zeta^{\nu} - (\mu \leftrightarrow \nu)$$

$$
\begin{aligned}
R^{\bar\mu\bar\nu} &= \tfrac{1}{4}[\bar\zeta W_{k,m}g_w]^{\bar\mu}[\bar\zeta(F^{km}g_w - 2ig_w^2[W^k,W^m])]^{\bar\nu} + \tfrac{1}{4}[\bar\zeta W_k W_m g_w^2]^{\bar\mu}[\bar\zeta[W^k,W^m]g_w^2]^{\bar\nu} \\
&- \tfrac{i}{2}[\bar\zeta W_k g_w]^{\bar\mu}[\bar\zeta(2W^k{}_{,m}W^m g_w^2 - 2W_m W^{k,m}g_w^2 - W_l{}^{,k}W^l g_w^2 - W_l{}^{,l}W^k g_w^2 + W_l W^{l,k}g_w^2 \\
&+ W^k W_l{}^{,l}g_w^2 - 2iW_m W^k W^m g_w^3 + iW_{l,m}{}^{,l}g_w - iW^k{}_{,l}{}^{,l}g_w + iW_l W^l W^k g_w^3 + iW^k W_l W^l g_w^3))]^{\bar\nu} \\
&- 9c^2\tfrac{1}{l^4}\zeta^{\bar\mu}\zeta^{\bar\nu} - (\mu\leftrightarrow\nu)
\end{aligned}
$$

$$
\begin{aligned}
R^{m\nu} &= \tfrac{1}{2}c(\bar\zeta\zeta)\mathrm{Tr}(W_k g_w)[\mathcal{F}^{mk}g_w\zeta]^\nu - \tfrac{1}{2}c(\bar\zeta\zeta)[\mathcal{F}^m{}_k W^k g_w^2\zeta]^\nu \\
&- ic^2\tfrac{1}{l^2}(\bar\zeta\zeta)[W^m g_w\zeta]^\nu - ic^2\tfrac{1}{l^2}[\bar\zeta W^m g_w\zeta]\zeta^\nu + ic^2\tfrac{1}{l^2}(\bar\zeta\zeta)\mathrm{Tr}(W^m g_w)\zeta^\nu \\
&- \tfrac{1}{2}c[\bar\zeta W_k g_w\zeta][\mathcal{F}^{mk}g_w\zeta]^\nu - \tfrac{i}{4}l^2[\bar\zeta(\mathcal{F}^k{}_l\mathcal{F}^{ml}g_w^2 + \mathcal{F}^m{}_l\mathcal{F}^{kl}g_w^2)\zeta][W_k g_w\zeta]^\nu \\
&- \tfrac{1}{2}[(2W^m{}_{,k}W^k g_w^2 - 2W_k W^{m,k}g_w^2 - W_l{}^{,l}W^m g_w^2 - W_l{}^{,m}W^l g_w^2 + W_l W^{l,m}g_w^2 + W^m W_l{}^{,l}g_w^2 - \\
&2iW_k W^m W^k g_w^3 + iW_{l,k}{}^{,l}g_w - iW^m{}_{,l}{}^{,l}g_w + iW_l W^l W^m g_w^3 + iW^m W_l W^l g_w^3)\zeta]^\nu
\end{aligned}
$$

$$
\begin{aligned}
R^{m\bar\nu} &= \tfrac{1}{2}c(\bar\zeta\zeta)\mathrm{Tr}(W_k g_w)[\bar\zeta\mathcal{F}^{mk}g_w]^{\bar\nu} - \tfrac{1}{2}c(\bar\zeta\zeta)[\bar\zeta W_k\mathcal{F}^{mk}g_w^2]^{\bar\nu} \\
&+ ic^2\tfrac{1}{l^2}(\bar\zeta\zeta)[\bar\zeta W^m g_w]^{\bar\nu} + ic^2\tfrac{1}{l^2}[\bar\zeta W^m g_w\zeta]\zeta^{\bar\nu} - ic^2\tfrac{1}{l^2}(\bar\zeta\zeta)\mathrm{Tr}(W^m g_w)\zeta^{\bar\nu} \\
&- \tfrac{1}{2}c[\bar\zeta W^k g_w\zeta][\bar\zeta\mathcal{F}^m{}_k g_w]^{\bar\nu} + \tfrac{i}{4}l^2[\bar\zeta(\mathcal{F}^k{}_l\mathcal{F}^{ml}g_w^2 + \mathcal{F}^m{}_l\mathcal{F}^{kl}g_w^2)\zeta][\bar\zeta W_k g_w]^{\bar\nu} \\
&+ \tfrac{1}{2}[\bar\zeta(2W^m{}_{,k}W^k g_w^2 - 2W_k W^{m,k}g_w^2 - W_l{}^{,l}W^m g_w^2 - W_l{}^{,m}W^l g_w^2 + W_l W^{l,m}g_w^2 + W^m W_l{}^{,l}g_w^2 - \\
&2iW_k W^m W^k g_w^3 + iW_{l,k}{}^{,l}g_w - iW^m{}_{,l}{}^{,l}g_w + iW_l W^l W^m g_w^3 + iW^m W_l W^l g_w^3)]^{\bar\nu}
\end{aligned}
$$

$$
\begin{aligned}
R^{\mu\bar\nu} &= \tfrac{1}{l^4}\Big(20c - 44c^2(\bar\zeta\zeta) + 44c^3(\bar\zeta\zeta)^2\Big)\delta^{\mu\bar\nu} + \tfrac{1}{l^4}\Big(18c^2 - 48c^3(\bar\zeta\zeta)\Big)\zeta^{\bar\nu}\zeta^\mu \\
&+ \tfrac{1}{4}l^2[\bar\zeta W_k g_w]^{\bar\nu}[W_m g_w\zeta]^\mu[\bar\zeta(\mathcal{F}^k{}_l\mathcal{F}^{ml}g_w^2 + \mathcal{F}^m{}_l\mathcal{F}^{kl}g_w^2)\zeta] \\
&+ \tfrac{1}{2}[\bar\zeta W_{k,m}g_w]^{\bar\nu}[\mathcal{F}^{km}g_w\zeta]^\mu + 2c^2\tfrac{1}{l^2}(\bar\zeta\zeta)[\bar\zeta W_k g_w]^{\bar\nu}[W^k g_w\zeta]^\mu \\
&+ c^2\tfrac{1}{l^2}\Big([\bar\zeta W^k g_w]^{\bar\nu}\zeta^\mu + \zeta^{\bar\nu}[W_k g_w\zeta]^\mu\Big)\Big([\bar\zeta W_k g_w\zeta] - (\bar\zeta\zeta)\mathrm{Tr}(W_k g_w)\Big) \\
&+ \tfrac{i}{2}c\Big([\bar\zeta W_m g_w]^{\bar\nu}[\mathcal{F}^{km}g_w\zeta]^\mu - [\bar\zeta\mathcal{F}^{km}g_w]^{\bar\nu}[W_m g_w\zeta]^\mu\Big)\Big([\bar\zeta W_k g_w\zeta] - (\bar\zeta\zeta)\mathrm{Tr}(W_k g_w)\Big) \\
&+ \tfrac{i}{2}c(\bar\zeta\zeta)[\bar\zeta W_m\mathcal{F}_k{}^m g_w^2]^{\bar\nu}[W^k g_w\zeta]^\mu + \tfrac{i}{2}c(\bar\zeta\zeta)[\mathcal{F}^k{}_m W^m g_w^2\zeta]^\mu[\bar\zeta W_k g_w]^{\bar\nu} \\
&- \tfrac{i}{2}[\bar\zeta[W_k,W_m]g_w^2]^{\bar\nu}[W^{k,m}g_w\zeta]^\mu + \tfrac{1}{2}[\bar\zeta W_k W_m g_w^2]^{\bar\nu}[[W^k,W^m]g_w^2\zeta]^\mu \\
&- \tfrac{i}{2}[\bar\zeta W_k g_w]^{\bar\nu}[(2W^k{}_{,m}W^m g_w^2 - 2W_m W^{k,m}g_w^2 - W_l{}^{,k}W^l g_w^2 - W_l{}^{,l}W^k g_w^2 + W_l W^{l,k}g_w^2 + W^k W_l{}^{,l}g_w^2 - \\
&2iW_m W^k W^m g_w^3 + iW_{l,m}{}^{,l}g_w - iW^k{}_{,l}{}^{,l}g_w + iW_l W^l W^k g_w^3 + iW^k W_l W^l g_w^3)\zeta]^\mu \\
&+ \tfrac{i}{2}[W^k g_w\zeta]^\mu[\bar\zeta(2W_{k,m}W^m g_w^2 - 2W_m W_k{}^{,m}g_w^2 - W_{m,k}W^m g_w^2 - W_m{}^{,m}W_k g_w^2 + W_k W_m{}^{,m}g_w^2 + \\
&W_m W^m{}_{,k}g_w^2 - 2iW_m W_k W^m g_w^3 - iW_{k,m}{}^{,m}g_w + iW_{m,k}{}^{,m}g_w + iW_k W_m W^m g_w^3 + iW_m W^m W_k g_w^3)]^{\bar\nu}
\end{aligned}
$$

Appendix B

Mathematica Code Documentation

The majority of the work done in this thesis was completed by using Mathematica. The collection of Mathematica notebooks that were used is available from the UTAS library digital repository. This Appendix is the documentation for that code, outlining how it is used and the general structure and purpose of the code. It is broken up into three sections, General Relativity with two coordinates which covers the work from Chapter 5, General Relativity with one coordinate which covers the work from Chapter 4 and finally field expansions which covers the work from Chapter 2. The reason this appendix is arranged backwards is due to the development process involved in producing the code. The first year of my PhD was working on the field expansions, however this was done while I was learning Mathematica and it is not the core of this thesis, so it will be covered last. The code to do general relativity was originally conceived in the one coordinate case, when that was completed it was reworked and upgraded to be able to handle the two coordinate case. These improvements meant that it was actually easier to scale back the two coordinate case down to the one coordinate case rather than use the original code. As a result of this the one coordinate code included here is a modified version of the two coordinate code, and so is included after it.

It should also be noted that this code was developed alongside the work on the thesis, so that much of it was included as an extension or work around to the previous code, rather than as part of an overall grand structure. Like most code for academic purposes it isn't exactly written in the best fashion, nor is it the most efficient code, but it does work and produces the necessary results. It is hoped that if the reader wishes to continue on with this work they can make use of the code provided, possibly modifying or using sections as they see fit. It is written in a fairly modular fashion, so some parts should be able to be upgraded without affecting others.

B.1 General relativity with two coordinates

This code produces the results seen in Chapter 5 and Appendix A. It represents the bulk of the work done to produce this thesis, with countless revisions and modifications over three years to produce it. The core of this is GRSU2.nb, which contains the majority of the code. The other notebooks are there to make use of what is available in that first notebook.

B.1.1 GRSU2.nb

This is the core of the GR with two coordinates code. The cells are mostly hidden, with code grouped based on function. This is done because otherwise the code would be too long to work with. The cells of code can be opened and closed via Alt C P O in Mathematica 8. It should be noted that every function has a `Clear` before it is defined. This is useful when working on developing the functions, as old definitions do not get in the way of new ones. Mathematica 8 sometimes seems to ignore this though, and a full restart of the kernel is required, via `Quit[]`.

Setup code

This is the setup cell, it also defines how noncommutative multiplication works between the property coordinates ζ.

`tensors` defines which symbols are to be considered tensors. Originally it was intended that this would be significant, but in the end it turned out the only thing it was necessary for was the output function `texform`.

`ncsimp` is the function to reduce a non commutative expression. So for instance `ncsimp`[ζ[1]**0+ζ[2]**2], will return 2ζ[2]. `dist`, `ncfactor` and `noncomQ` are all helper functions for `ncsimp`, which is the only function that the end user should apply from this section. Note that this version of `ncsimp` also requires helper functions from a later section, namely `greaterlist` and `expidlist`.

The inbuilt function `NonCommutativeMultiply` is used as the basis for non commutative multiplication. This was primarily because it worked with the inbuilt function `Distribute`. Expressions involving multiple ζ will automatically rearrange into a set order, so for instance $\zeta[1] * *\zeta[2]$ outputs as $-\zeta[2] * *\zeta[1]$. Expressions with more than one of the same ζ evaluate to zero.

If the reader is intending to continue with this work, it may be a good idea to write a custom function to deal with non commutative expressions. The code using `NonCommutativeMultiply` is functional and was sufficient for this work, but issues arose when attempting to tidy up expressions from the two coordinate case, resulting in some messy code. There also seems be some recursion depth issues in the interaction between `NonCommutativeMultiply` and the custom derivative operator I wrote when acting on large expressions.

Texform output code

One of the critical breakthroughs I had in learning Mathematica was the concept of separating the output form of an expression from the internal working Mathematica version. Take tensors for example, you want them to look similar to how you would write them, say g_{mn}. Mathematica can do one index nicely using a subscript but there are problems with two, just having m n will make a subscript of m times n. mn with no gap will be a new symbol unrelated to m or n. One option is to instead make a list, so the tensor becomes $g_{\{m,n\}}$. This however doesn't look very neat and is a pain to work with functionally, it is a poor compromise on two fronts.

The solution to this is to separate out the functional expression from the nice looking output. Mathematica has an inbuilt `ToString[ ... ,TeXForm]`, which will give the Latex code to reproduce an expression. This doesn't cover what is needed for a custom output function though, so I wrote my own. The code to do this should be applicable to a wide range of projects in Mathematica, anywhere that a custom output to Latex is required. The tensor from before now becomes `g[d[m],d[n]]`, this doesn't look as nice but it is significantly more useful in terms of functionality than any of the previous suggestions. The `d`'s can be replaced with `u`'s to make the index `up` rather than `down`. We can also use the function from this section, `texform`, as follows: `texform[g[d[m],d[n]]]` to produce the following string: `g{}_{m}{}_{n}` which displays in Latex as g_{mn}.

If we apply `texform` to the expression `a+2(b+c)` the resulting string is `(a+2 (b+c))`. If we don't want to have the brackets around the whole thing then the function `totex` can be used as a wrapper to `texform` to remove these outer brackets from the expression. So for example `totex[a+2(b+c)]` returns `a+2 (b+c)` instead. This doesn't seem like it would save much effort, but I got sick of removing those brackets over and over when working with output.

`totexline` is another useful function, if you give it a sum of terms it will give the output with each term separated onto its own line. So for instance `totexline[a+b+c]` returns the string `a\\+b\\+c\\`. You can then also use the find-replace function in Latex to replace the \\ with a \\ followed by a new line if you want to make the Latex more readable. `totexline` is especially useful when trying to deal with very large/complicated expressions as it makes it easier to see what is going on.

The beauty of doing output like this is once it is setup like we have done, setting custom output formats for any type of expression is very easy. So for instance we can define `texform[ζsum]:= "(\\bar\\zeta \\zeta)"` so that the expression `ζsum` outputs as `(\bar\zeta \zeta)` which appears in Latex as $(\bar{\zeta}\zeta)$. Note the double backslashes, as backslashes are used as escape characters in Mathematica two of them are required to produce one in the output string.

`texplus`, `texplusline`, `bartex`, `texindex`, `getvectorelements` and `getmatrixelements` are all helper functions for `texform` to deal with various types of expression. The functions that should be used by the end user are `texform`, `totex` and `totexline` depending on what is required, though use of `texform` directly could be dropped in favour of `totex`.

Basic simplification rules

This section contains a series of rules and functions to simplify the expressions that arise in this work. The rules are the replacement lists, like for instance `flatspaceassumptions`, while the functions are like `deltasimp`. Rules are applied via the replace function, which can be expressed in shorthand as `exp /. flatspaceassumptions`, or for a repeated replace as `exp //. flatspaceassumptions`. The functions can be applied via a similar postfix notation, say `exp // deltasimp`. Writing a series of rule or function applications like this makes it much easier to see what is being performed on the expression. Note that care has to be taken if a long series of functions and rules are applied at once, as the order in which they apply can become muddled unless enforced by intermediate expressions or brackets. There are examples of using these simplification rules in the other notebooks. We will now cover some of the more useful simplification tools.

`flatspaceassumptions` is a replacement rule that removes any space-time derivatives of the standard space-time metric. This is useful when trying to simplify a complex expression and the space-time curvature isn't relevant.

Einstein summation convention is used extensively throughout this work, and many of the expressions end up with multiple indices that are being summed over. To help simplify and match up terms that are the same `dummyswap` is used to swap these indices to a standard set of indices of the same type, a label with a subscript. Note that `dummyswap` has multiple arguments, the first is the expression that is being modified, the second is the list of dummy indices to be swapped and the last is the label to be used. This function is sort of obsolete, in that we started using standardised dummy variables from the start instead of using this function.

`deltasimp` is a function used to simplify expressions involving products with delta functions. `deltasum` is a replacement rule used to simplify delta functions that are being contracted over themselves. `propdeltasimp` does the same thing as `deltasimp` except to property indices only. The difference is specified by the use of δp rather than δ. `fulldeltasimp` is a function that applies `deltasimp` over and over until the expression stays the same.

`matconvert` is a rule used to take gauge field matrix elements and to combine them into matrix products. For example it takes ${A_m}^{\mu\bar{\nu}}{A_n}^{\nu\bar{\rho}}$ to $(A_m A_n)^{\mu\bar{\rho}}$. Gauge matrices are placed inside a wrapper function labeled `matA`, where the first argument is a list of the matrices contained to maintain their ordering. The last two arguments are the indices associated with the overall matrix product.

`matsumconvert` is a rule to take sums of these matrix products with the same indices and group them together. So for instance it takes $(A_m A_n)^{\mu\bar{\rho}} + (A_n A_m)^{\mu\bar{\rho}}$ to $(A_m A_n + A_n A_m)^{\mu\bar{\rho}}$. It must be used after `matconvert` and places the `matA`'s into another wrapper function called `matAsum`. The first argument of this wrapper is the sum of `matA`'s with their indices stripped. The last two arguments are the indices from the `matA` contained in the wrapper. There is a variation of `matsumconvert` provided called `nonmatsumconvert`, which is used if you wish to skip applying the summation simplification of `matsumconvert`.

An expression may end up with a series of dummy variables of a standard form, say sums over s_1, s_2, s_4, skipping some numbers. `relabeldummy` is used to relabel those indices and to not skip any numbers. The arguments are the expression to be modified and the dummy variable being relabeled. To apply a function with multiple arguments like `relabeldummy` using postfix notation the following syntax can be used: `exp // relabeldummy[#,s]&`. The `#` is a wild card that gets replaced by `exp`, `s` is the label of the dummy index and the `&` is what matches the function to the `#`. The Mathematica documentation covers this sort of syntax well. There is a second version of `relabeldummy` that takes a third argument, this is used to set what number the indices start counting from. This is useful when attempting to simplify parts of an expression separately and do not want to have an overlap of dummy index labeling.

`propint` is a function that drops any terms that do not contain exactly 2 ζ and 2 $\bar{\zeta}$. This is essentially equivalent to integrating over property.

`tracecon` is a rule that integrates over property on expressions that contain the correct number of ζ and $\bar{\zeta}$, as well as possibly some gauge field matrices. It then converts these to the correct trace expressions. Note that it works on `matAsum`, which means that `matsumconvert` or `nonmatsumconvert` has to be applied first.

`fullsimp` applies a series of rules to simplify an expression without needing to manually apply each one. Note that it has a factorial computational time in the number of dummy indices due to the use of `fixdummyorder`, which hasn't been covered yet. It also assumes you are using s for space-time dummy indices and α for property dummy indices. There is a version of `fullsimp` that also takes another argument, this is the number that is used by `relabeldummy` to start where the dummy indices are labeled from.

`quicksimp` is a function similar to `fullsimp`, except it uses `quickswap` instead of `fixdummyorder`. `quickswap` will be introduced in the next section.

`ζsumreplace` is a replacement rule that collects up SU(2) invariants $\bar{\zeta}\zeta$. Such invariants commute and so can be removed from the non commutating parts of the expression.

`formζprod` is a rule that takes a product like $\zeta^{\bar{\mu}} X^{\mu\bar{\nu}} \zeta^{\nu}$ and converts it to $\bar{\zeta}X\zeta$, where X is some combination of gauge matrices. It again uses `matAsum`, so `matsumconvert` or `nonmatsumconvert` has to be applied first.

`ζXζsums` is a rule similar to `matsumconvert`, it takes expressions like $\bar{\zeta}X\zeta + \bar{\zeta}Y\zeta$ and converts them to $\bar{\zeta}(X+Y)\zeta$.

`traceconvert` is a rule used to simplify expressions like ${A_m}^{\mu\bar{\mu}}$ to $\mathrm{Tr}(A_m)$

`Fsimp` is a rule to simplify down gauge matrices into the abelian field tensor F and anti commutators $[A_m, A_n]$ in an attempt to produce expressions containing the non abelian field tensor $\mathcal{F}$.

`tracesum` is a rule to simplify sums of traces, so that Tr(A)+Tr(B) gives Tr(A+B).

Dummy swapping code

When using Einstein summation convention in general relativity, calculating the Ricci tensor and scalar produces a large number of terms many of which contain a number of summed indices. These internally summed indices, which we will call dummy indices, can have their labels swapped around without affecting the resulting expression. To simplify these expressions it is necessary to have some method of recognising patterns and grouping terms that are similar. The method we use is to first give all indices of a specific type a standard form. We chose to use s_i for space-time indices and α_i for property indices. We then need all the terms to have the same set of dummy indices, this is achieved by using `relabeldummy`. With this done we want to match terms that have the same pattern of dummy indices, though they may be in a different order. For instance $A_{s_1} {g^{s_1 s_2}}_{,s_2}$ is the same as $A_{s_2} {g^{s_2 s_1}}_{,s_1}$, but Mathematica does not instantly recognise this. One way to tackle this problem would be do go through and compare every term against every other term and see if they match up. This however would take N^2 time, where N is the number of terms which is of the order ~ 1000.

A better way is to go through each term individually and re arrange the dummy indices into a canonical order, this is what the functions `fixdummyorder` and `quickswap` do.

`fixdummyorder` has two arguments, one for the expression and the other for the dummy index to be re arranged, so either s or α. It assumes the dummy indices are in the form of s_i, with a label and a numerical subscript. It works by going to each term in a sum individually and determining what dummy indices of the indicated label are present. It then generates a list of every possible permutation of those indices and applies each of those permutations to the expression one at a time. To determine which permutation to use it converts each permuted expression into a string and then that string into a vector and then compares element by element along the vector to find the "minimum" list. After going through every permutation of the dummy indices the version with the "minimum" permutation is returned. This is done for every term in the sum separately, resulting in a computational time of $NS!$ where S is the number of dummy indices. `fixdummyorder` will always put an expression containing dummy indices into a canonical order, but due to the $S!$ time dependence takes a long time for expressions involving $\sim$ 7+ dummy variables. To deal with this `quickswap` was developed to be computationally faster, even if it wasn't 100% reliable.

`quickswap` is a function designed to be less computationally intensive than `fixdummyorder`. It works by simply taking the list of dummy variables and then sorting it and swapping the dummy indices into that order. It is far faster than `fixdummyorder` but sometimes fails to fully simplify expressions because swapping the indices can also swap around the order of a product. This code was developed because `fixdummyorder` was taking too long to complete on longer expressions. Using `quickswap` first to cancel some terms out and then applying `fixdummyorder` at the end results in a much quicker computational time.

Contraction over property sums

This code is designed to be used when all sums are explicitly expanded out in terms of property indices. So instead of writing $\zeta^{\bar{\mu}}\zeta^{\mu}$, this would be given as $\zeta^{\bar{1}}\zeta^{1}+\zeta^{\bar{2}}\zeta^{2}$ for the 2 coordinate case. `contractpropertysums` then looks to undo this expansion by searching for terms that are identical except for increased indices in the same spot and then replacing those terms with single term with a summation index instead of explicit numerical indices. It takes 4 arguments, the first is the expression being contracted, the next two are the types of sums it should look for. In the context of this work these can be ζ, ζ`b`, `u`, `ub`. Make sure the pairing you put in makes sense, there is no check here for that, so for instance ζ should be paired with ζ`b` or `ub`, never itself or `u` as that wouldn't make sense. The last term is the dummy index it should use to replace these terms with, so usually α. It will search the expression and use the next available number after the highest index present, i.e. if a sum is performed with the dummy variable as α and there are currently dummy indices α_1 and α_3 present it will use α_4. As an example of this code in action,

```
ζb[1]ζ[1] + ζb[2] ζ[2] // contractpropertysums[#, ζb, ζ, α] &
```
$= \zeta$`b[`α_1`] ζ[`α_1`]`.

This code is semi-obsolete now, as the we moved away from explicit index expansions to working with summation indices the whole way through. It may still be useful in the future for other projects though.

Metric and inverse metrc

This cell contains the definition of the metric and its inverse. The metric tensor is setup in two steps, first `G[m,n]`, `G[m,ν]`, `G[m,ν̄]` and `G[μ,ν]` are defined as in Chapter 5. Note that α_i is used as the summation index, with great care taken to ensure that there is only one pair of each. Also the delta function used in `G[μ,ν]` is δ`p`, indicating it is a delta function over property coordinates only.

Using the definition of `G` the function `metric` is then defined, which is then used later as the metric tensor. `metric` has three arguments, the first two specify the coordinates, the last specifies the indices for the property sums. There are three options when entering a coordinate, `even`, `odd` and `oddc` corresponding to x, ζ and $\bar{\zeta}$ like coordinates respectively. The list of indices is entered as a list of numbers of length `metpropind`, which is 4. This argument is necessary as there will be expressions where the metric is multiplied by itself, to avoid a repetition of summation variables there has to be a way to specify the summation variables to be used. The function `metric` also takes care of the symmetry properties of the metric, so for instance `G[μ,n]` does not need to be defined explicitly. The syntax for using `metric` is as follows: `metric[even[l],even[p],{1,2,3,4}]` or `metric[oddc[λ],odd[ρ],{5,6,7,8}]` and so on. The variables can be labeled however you like, but we stick to the convention defined at the start of Chapter 3.

The inverse metric is setup in a similar way to the metric, first `invG` is defined as in Chapter 5. The only significant difference here is that there are now space-time summation variables as we have chosen to only have covariant gauge matrices ${A_m}^{\mu\bar{\nu}}$ rather than also including $A^{m\mu\bar{\nu}}$. This choice was made primarily to help with simplification. We have written routines that can deal with swapping dummy indices around, so having all the gauge matrices covariant and then a series of inverse space-time metrics g^{mn} makes everything easier. This difference is important though when it comes to the definition of `invmetric`, this function is based on `invG` similar to how `metric` was defined, but has four arguments instead of three. The first two are the coordinates, with the same `even`, `odd` and `oddc` system. The next is a list of numbers of length 2 to label the space-time summations present. The last is a list of numbers to label the property summations of length 3. So to call the inverse metric the syntax is `invmetric[even[m],even[n],{1,2},{1,2,3}]`.

Following the definitions of the metric and its inverse is some commented out code that finds $G^{LM}G_{MN}$ and $(-1)^M G_{LM}G^{MN}$ to test that the inverse metric is correct. This can be commented in if testing or modification of the metric and its inverse is to be done.

Standard GR versions of Ricci tensor/scalar etc

This is where the standard space-time versions of the Christoffel symbols, Riemann curvature tensor, Ricci tensor and Ricci scalar are done. What is important here is that the formulas used take exactly the same form as those used to get the full extended versions, so that it becomes easy to identify the standard GR components of our extended expressions. Most of them are fairly straight forward, with perhaps the exception of the raised Christoffel symbol which takes an extra argument for the internal summation index with the metric. Here is a table of the function names and the corresponding symbol:

`connG[m,n,l]`	${\Gamma^{[g]}}_{mnl}$
`ΓG[m,n,l,k]`	${\Gamma^{[g]}}_{mn}{}^{l}$
`riemannG[m,n,l,p]`	$R^{[g]m}{}_{nlp}$
`covriemannG[m,n,l,p]`	${R^{[g]}}_{mnlp}$
`ricciG[m,n]`	${R^{[g]}}_{mn}$
`ricciscalarG`	$R^{[g]}$
`contraricciG[m,n]`	$R^{[g]mn}$

Derivative operator

In this section the grading function and custom derivative operator are defined. The grading function `gr` is quite simple, since `even`, `odd` and `oddc` are specified. The primary use of this function is for the sign factors, say `(-1)`$^{\texttt{gr[M]}}$.

The derivative operator `deriv` is defined as a series of rules that allows it to handle any type of derivative expression that turns up in the work. It can do both space-time and property coordinate derivatives, correctly applying the product rule for each case. The first argument is the expression the derivative is being taken of, the second is the derivative to be applied. The syntax for using this function is then `deriv[expression,p]` for a space-time derivative or `deriv[expression,ζb[`μ`]]` for a derivative with respect to $\zeta^{\bar{\mu}}$. The derivative operator is actually defined inside another function called `defderiv`, this means the derivative operator can be effectively turned off by using `Clear[deriv]` and then back on again using `defderiv`. This was done because sometimes infinite recursion was occurring when simplifying an expression, which was stopped by temporally disabling `deriv`.

It was also necessary to define a wrapper function for the derivative operator `nderiv`, which takes derivatives of the form `deriv[expression,even[b]]` and changes `even[b]` to `b`. To use this simply replace `deriv` with `nderiv` in expressions that involve substitution of variables with their grading tags still attached.

Full GR connection coefficients

The extended Christoffel symbols are defined here. `fullsumlist` is used when expanding explicitly over ζ^1, ζ^2 etc, it isn't used in this version of the code. `indexsumlist` is used when summation indices are used. It takes two arguments, one for the space-time index and one for the property sector index.

The last line of code increases the recursion limit as it was being reached, a custom version of NonCommutativeMultiply could have possibly avoided this.

All of the following notebooks make use of the code contained in GRSU2.nb to calculate the various expressions required for this thesis, GRSU2.nb must be executed first before using these other notebooks. There is some repeated code, as many routines were developed while working on a specific problem and then integrated into GRSU2.nb once they were sufficiently tested.

B.1.2 frame.nb

This notebook was used when testing the frame vectors and inverse frame vectors. It defines `frame` and `invframe` in a similar manner to how `metric` and `invmetric` work. It also defines `eta` and `inveta` as $\mathcal{H}_{MN}$ and $\mathcal{H}^{MN}$ respectively, which is the extended Minkowski metric. G_{MN} and G^{MN} are then calculated from these frame vectors, as well as multiplying the frame vector by its inverse to ensure it is correct.

B.1.3 christoffelworking.nb

This notebook was used to produce the list of Christoffel symbols seen in Appendix A. Simply change the first line to the required coordinates, run it and then grab the resulting Latex output string.

B.1.4 palatini.nb

This notebook calculates the Palatini form of the Ricci scalar. `exp` is the extended Palatini form, which is simplified and then multiplied by $\sqrt{G_{..}}$ to become `exp4`. This is then further simplified and has the standard GR part subtracted to become `exp10`. The rule `cyctracesimp` is defined to re arrange the traces to help simplify the expression further. There are two versions given, the second has the replacements for the scalar expectation values that our current paper in preparation uses.

B.1.5 riccitensor.nb

This is where the Ricci tensor components can be pre-calculated, so that other notebooks that need to use them can do so without excessive computational time. First the extended Ricci tensor is defined as the function `Ricci`. Then the 6 independent components of the Ricci tensor are calculated, simplified and used to setup the function `precalcRicci`. The standard space-time versions of the Ricci tensor, raised Ricci tensor and Ricci scalar are also calculated.

B.1.6 raisedriccitensor2.nb

The list of raised Ricci tensor components R^{MN} in Appendix A was determined using this notebook. First the function `raisedricci` is defined as the extended Ricci tensor R^{MN}. Notice that in the definition rather than using `ncsimp` and `precalcRicci` instead `tncsimp` and `TprecalcRicci` are used. This is done because the simplifications are very computationally intensive, and a temporary version of the functions is used so that the simplifications can be done in smaller parts. You will notice later on these temporary functions are changed back to their functional forms. The functions `symsimplify` and `symsimplifyζXζ` are used to simplify R^{mn} by removing the terms that have n appearing before m, since we know R^{mn} is symmetric. `graise` is used to raise indices with the space-time metric. Since we are dropping the derivatives of the metric here this can be applied freely. `sF2simp` matches up combinations of F_{mn} and $[A_m, A_n]$ into $\mathcal{F}_{mn}$. `matAcount` is used to place in the correct factors of g_w as required, since these are not included in the metric originally. The six independent components of R^{KM} are then calculated one at a time in the rest of the notebook. This notebook requires that riccitensor.nb is run first as it uses `precalcRicci`.

B.1.7 precalcraisedriccitensor.nb

Similar to riccitensor.nb, this notebook pre calculates the Raised Ricci tensor components for use in the field equations. `raisedricci` is defined as the extended raised Ricci tensor and then this is used to calculate each of the three necessary components of R^{KM}. These are then simplified and used to define `precalcRaisedRicci`, which is similar to `precalcRicci`. The Ricci scalar is also calculated here for use in the field equations. This notebook also requires that riccitensor.nb is run first.

B.1.8 ricciscalar.nb

This notebook calculates the Ricci scalar and then the Lagrangian to check against the result from palatini.nb. Note that riccitensor.nb must be run first.

B.1.9 fieldequations.nb

The field equations for the space-time metric are calculated in this notebook. `factor1` is $\sqrt{G_{..}}\delta G_{mn}/\delta g_{mn}$, which is multiplied by $R^{mn} - \frac{1}{2}G^{mn}R$ in `exp1`. This is then simplified and the component proportional to $R^{[g]mn} - \frac{1}{2}g^{mn}R^{[g]}$, which is defined as `einsteinG`, is subtracted. This is simplified further by making use of the cyclic properties of the trace to get the final result. This notebook requires precalcraisedriccitensor.nb to be run first.

B.1.10 maxwell up.nb

Similar to fieldequations.nb, this notebook calculates the field equations for the gauge field. `varG` is the variation of the metric G_{MN} with respect to ${A_p}^i$. Note that instead of writing τ_i for the basis matrix it is implemented as A_τ as this was easier. The calculations are broken down into parts via temporary functions again, notice the use of `ncsimp1o`, `ncsimp2o` and `ncsimp3o` in `exp`. Executing `ncsimp1o` $\to$ `ncsimp` took over 30 minutes, though the other steps are faster. Cyclic trace simplifications are then used to get the final result, which is broken up into two parts, where $c_1 = 0$ or $c_3 = 0$. Some additional work by hand was required to get the result in the Chapter 5 with all the terms containing c_1 canceling out. This notebook requires precalcraisedriccitensor.nb to be run first.

B.2 General relativity with one coordinate

This is the code used to generate the results from Chapter 4, it is based on the code for the two coordinate case. The primary change is the fact that there are no longer property indices to worry about so that instead of writing μ, $\bar{\mu}$, etc we can just use ζ or $\bar{\zeta}$.

B.2.1 GRemag.nb

This is the core of the code for the 1 coordinate case, it needs to be executed before the other notebooks are used. In general it is a much simpler version of GRSU2.nb, much of the same functionality is retained with similar syntax. Some parts have not been trimmed to completely get rid of references to property indices, but this does not effect how they function.

The largest difference in functionality between this code and GRSU2.nb is in using the metric. The metric is defined in a similar manner, but since there are no property indices the list of them is no longer necessary. Thus `metric` has two arguments and `invmetric` has three, as `invmetric` still requires the two space-time indices. The tags `even`, `odd` or `oddc` are also no longer necessary, since ζ and $\bar{\zeta}$ are used directly the syntax becomes `metric[m,ζ]` and so on.

GRemag.nb also includes the definitions for the extended Ricci tensor as well as pre-calculations for the Ricci tensor and scalar. These are simple and fast enough to do in this case that they could be included in this notebook rather than using a separate one.

B.2.2 working.nb

This notebook contains examples of code working with GRemag.nb. The Ricci scalar is calculated first as `rscalar2`, then this is used to find the Lagrangian in `lag4`. Note the space-time part of the Lagrangian is subtracted to form `lag3`. Part of the raised gravitational field equations is then considered for the rest of the notebook.

B.2.3 variation.nb

This notebook looks at the variation of the Lagrangian with respect to the raised metric tensor, producing the lowered gravitational field equations. `varG` is defined like in the two coordinate case. The standard GR component is subtracted off to form `exp6` and then Maxwell part is subtracted off to form the remaining result `exp7`.

B.2.4 maxwell up.nb and maxwell down.nb

These notebooks calculate the raised and lowered versions of the field equations for the gauge field. `varG` is defined like usual and then the resulting variation is calculated. `Fdecomp` is defined to break up the field tensor F_{mn} into $A_{n,m} - A_{m,n}$, this is done so that we can use an write expression that we match against the result. `testexp` is defined, expanded out, simplified and then matched against the result and found to be identical.

B.3 Field expansions

This is the code used to generate the results in Chapter 2, it was written in the first year of my PhD as I was learning Mathematica. The notebooks and other documents can be used independently, copies of the necessary results are in the corresponding notebooks. The non commutative algebra for the property coordinates ζ is handled by a package called NCAlgebra. This works in conjunction with a function `zeta` which has two arguments, the first is the list of property indices, the second is the list of conjugate property indices. So for instance $\zeta^1\zeta^3\zeta^{\bar{2}}$ is entered as `zeta[{1,3},{2}]` and then displays in Mathematica as $\zeta^{\{1,3\}}_{\{2\}}$. `SNC` sets symbols to be non commuting in NCAlgebra, `SetCommutative` sets variables to be commuting. NCAlgebra also operates using `NonCommutativeMultiply` and has its own functions to simplify expressions, like `NCExpand` which can be shortened to `NCE`.

B.3.1 gen.m

This is matlab code that is used to produce the spreadsheet SU(5).xls. It goes through and generates all the possible terms in field expansions for a given (p, q) pair, i.e. number of ζ and $\bar{\zeta}$, as well as their corresponding charges. The user modifies zeta which is the number of

ζ and barzeta which is the number of $\bar{\zeta}$. The output is a comma seperated variables file with the zeta combinations and the corresponding charges. This code does not take into account the selfduality condition.

B.3.2 SU(5).xls

This spreadsheet lists the particle content of the fermionic field expansion and the corresponding charges. Charges are listed in units of 1/3 the electronic charge, i.e. the range shown of -6 to 6 is actually -2 e to 2 e. The headings above each set show the (p, q) pairing, and then there is a 1, -1 or a 0 to indicate whether each individual coordinate is present. 1 indicates ζ is present, a minus one indicates $\bar{\zeta}$ is present and a 0 indicates neither is present OR a pair $\bar{\zeta}\zeta$ is present. The pairs can be spotted via the (p, q) content. So for instance the first row on the second set of data: under `zeta^3 \bar{zeta}^0` is 1,1,1,0,0. This corresponds to a term of type $(3, 0)$, and specifically to ζ^{012}. The anti-selfduality condition was applied manually, for terms of type $(3, 2)$ or $(4, 1)$ those removed by anti-selfduality are in bold and are not included in the count of charges.

B.3.3 ferm2.nb

This notebook determines the mass matrices for the standard model fermions present in our model, namely neutrinos, electrons, up quarks and down quarks and their generations. There are a few operators that are defined to assist with this, `dual` finds the dual of a `zeta` term, `asd` finds the anti self dual combination, `conj` takes the charge conjugate and `aj` takes the adjoint.

The Higgs like field content is entered as `phi`, and then each of the other fermionic superfields is entered similarly. The Yukawa term is expanded out and the mass matrices are produced from this, the results are listed at the end. To select which fermion field to target, the variables `psi` and `particles` need to be changed accordingly. The current setting is to look at the red up quarks.

B.3.4 bosL.nb

This is where the conditions on the expectation values are determined. The expectation value of the Higgs superfield is entered as `phi` and then Φ^2, Φ^3 and Φ^4 are calculated and used to form the spontaneous symmetry breaking Lagrangian. Partial derivatives with respect to each of the expectation values are taken and the resulting set of equations is given as `conds`.

B.3.5 solvesystem.nb

This notebook details some of my attempts to find numerical solutions to this system. The conditions on expectation values as well as the mass matrices are included at the start. I then attempted to solve the system of conditions numerically and then found the resulting eigenvalues of the mass matrices. Several functions to help with this process were used, each of which takes in the three parameters from the Lagrangian, solves for the expectation values numerically and then outputs a result based on the eigenvalues of the mass matrices. `neutrinomass` returns the sum of the masses of the three lightest neutrinos, `upmass` finds the mass of the third lightest up quark, the top quark, and `leptonratio` finds the mass ratio of the lightest two leptons. These can also easily be modified, so for instance to find the mass of the lightest up quark `upmass` can be changed so the last line is `eigs[[1]]`.

Using these functions I did searches across the parameter space to try and find a sensible set of masses. The targets were usually trying to maximise the ratio of the up quark masses to the neutrino masses, since this needs to be order $\sim 10^{11}$ for the top quark mass divided by electron neutrino mass. In the middle of the notebook there are a series of plots, the x and y axis are the coefficients of Φ^3 and Φ^4 in the Lagrangian, while the vertical axis is the mass ratio being targeted. `ParallelTable` was used to allow for multi-cpu processing.

Below the plots are some searches by directly using the expectation values rather than the Lagrangian. `masstest` was used to check the ratio of the masses given a set of expectation values. A random parameter search was then done using `ParallelTable`. A random search was chosen because this allowed for the computational requirements to scale with the time available, it could be left running for several days.

B.3.6 eigenvalues.nb

This notebook details some attempts to analyse the properties of the matrices and their eigenvalues directly. The first part of this notebook contains the mass matrices and then a set of sliders that vary the expectation values of the Higgs field and the resulting mass Eigenvalues. The rest of the notebook involves attempts to work with the Eigenvalue equation for the neutrino mass matrix, but nothing usable came out of it.

B.3.7 selfduality.pdf and dualadj.pdf

These are notes written alongside the code, while some of the notation is a bit different they may prove useful to someone attempting to extend this work. Selfduality.pdf covers notation, duality and lists the possible states removed by anti selfduality. Dualadj.pdf covers the Mathematica implementation of the dual and adjoint operators, as well as a demonstration that they commute for all sets of odd numbers of coordinates, i.e. fermions.

Index of Terms

Index of Symbols